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(4) aids in the application of statistical techniques, such as nomographs, tables, work-sheet layouts, forms, and apparatus;
(5) critiques or reviews of significant studies involving the use of quantitative techniques.

The emphasis is to be placed on articles of type (1), in so far as articles of this type are available.

(Continued on the back inside cover page)

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Psychometrika

CONTENTS

ON ESTIMATION OF PARAMETERS IN LATENT STRUCTURE ANALYSIS 1
T. W. ANDERSON

SOLUTION OF THE PERSONNEL CLASSIFICATION PROBLEM WITH THE METHOD OF OPTIMAL REGIONS . 11
PAUL S. DWYER

THE MATHEMATICAL THEORY OF FACTORIAL INVARIANCE UNDER SELECTION 27
YRJÖ AHMAVAARA

THE SQUARE ROOT METHOD AND MULTIPLE GROUP METHODS OF FACTOR ANALYSIS 39
HARRY H. HARMAN

A SOLUTION FOR CASE III OF THE LAW OF COMPARATIVE JUDGMENT 57
RAYMOND H. BURROS AND W. A. GIBSON

THE PREDICTED AND OBSERVED EFFECT OF CHANCE SUCCESS ON MULTIPLE-CHOICE TEST VALIDITY . 65
LYNNETTE B. PLUMLEE

A SECOND-ORDER FACTOR ANALYSIS OF REASONING ABILITIES 71
LEONARD MATIN AND DOROTHY C. ADKINS

NOTE ON THE SELECTION OF A PANEL OF JUDGES SO AS TO MAXIMIZE PANEL EFFICIENCY 79
STEPHEN HARRISON, NORMAN H. ISHLER, AND ELSIE M. LAUE

CLYDE H. COOMBS, *A Theory of Psychological Scaling* 89
A Review by ALLEN L. EDWARDS

KENNETH L. BEAN, *Construction of Educational and Personnel Tests* 91
A Review by RUDOLPH G. BERKHOUSE

W. J. DIXON AND F. J. MASSEY, JR., *An Introduction to Statistical Analysis* 92
A Review by WILLIAM B. MICHAEL

BOOKS RECEIVED 94

VOLUME NINETEEN MARCH 1954 NUMBER 1

SUMMER STATISTICS SEMINAR

The University of Connecticut

The fifth Summer Seminar in Statistics will be held at the University of Connecticut, August 9 through 27, 1954. This Seminar is designed as a meeting place for statisticians with people in industry, commerce, or the physical sciences. Invited speakers will introduce topics which will later be subjects of general discussion. The plan is to have morning and afternoon sessions each of about two hours.

Registration and Lodging. Anyone interested is invited to attend for the day, week, or other period. The registration fee is $2.00 per week or $5.00 for the whole seminar. Lodging, either for one person, or a family, can be provided in a University dormitory at $1.75 per day per person. Meals will be available at a University cafeteria. A map of Storrs will be sent upon request. Transportation from train and air terminals to Storrs will be arranged by the Secretary if notified sufficiently in advance. Some grants for board and lodging for graduate students are available; application should be made to the Secretary. Further information may be obtained from the Secretary, Prof. Geoffrey Beall, Department of Statistics, University of Connecticut, Storrs, Connecticut.

August 9 through 13: Statistical Theories of Choice

Organizer, Prof. David Blackwell, Dept. of Mathematics, Howard University. Plans are to consider (1) the sufficiency principle, (2) the invariance principle, and (3) minimax, regret, Hodges-Lehmann, and Bayes' principles.

August 16 through 18: Applications of Statistics in Social Research

Organizer, Prof. Walter T. Federer, Dept. of Plant Breeding, Cornell Univ. Agri. Exp. Station.

August 19 through 21: Applications of Statistics in Meteorology

Organizer, Prof. Max Woodbury, Dept. of Statistics, Univ. of Pennsylvania.

August 23 through 27: (Joint session with Operations Research Society of America)

Organizers, Prof. John W. Tukey, Dept. of Mathematics, Princeton Univ., and Prof. George Kimball, Dept. of Chemistry, Columbia Univ.

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ON ESTIMATION OF PARAMETERS IN LATENT STRUCTURE ANALYSIS

T. W. ANDERSON*

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The latent structure model considered here postulates that a population of individuals can be divided into m classes such that each class is "homogeneous" in the sense that for the individuals in the class the responses to N dichotomous items or questions are statistically independent. A method is given for deducing the proportions of the population in each latent class and the probabilities of positive responses to each item for individuals in each class from knowledge of the probabilities of positive responses for individuals from the population as a whole. For estimation of the latent parameters on the basis of a sample, it is proposed that the same method of analysis be applied to the observed data. The method has the advantages of avoiding implicitly defined and unobservable quantities, and of using relatively simple computational procedures of conventional matrix algebra, but it has the disadvantages of using only a part of the available information and of using that part asymmetrically.

1. *Introduction*

Latent structure analysis is designed for investigations in which the fundamental data consist of sets of discrete statistical variables. The case of particular interest to social psychologists and sociologists is that in which each of a number of individuals responds positively or negatively to each of a number of items. In particular, these may be answers to dichotomous questions in a questionnaire. The analysis is based on a mathematical model for which the principal assumption is that the population may be classified into homogeneous sets such that in each set the responses on different items are independent in the probability sense. If the number of homogeneous sets is finite, we have what is called the latent class model. This model which is treated in this paper is given explicitly in the next section.

P. F. Lazarsfeld has written about latent structure analysis in Chapters 10 and 11 of (**1**). He and collaborators have also written a series of (mimeographed and dittoed) papers, "The Use of Mathematical Models in the Measurement of Attitudes," issued by the RAND Corporation (**2**). In these papers the rationale and purpose of the model are given as well as examples of the application of the analysis to data in psychology and sociology. The reader of this journal is also referred to the article by Green (**3**).

In an unpublished memorandum Lazarsfeld has indicated how the "latent parameters" corresponding to a particular item can be obtained from knowledge of the joint probabilities of the responses. The method is to

*Work supported by the RAND Corporation.

find the roots of a certain determinantal equation. In this paper the method is extended to obtain simultaneously the latent parameters corresponding to many items and the "latent class frequencies." The extension of the method involves the use of matrix algebra; in particular, use of vectors associated with the roots of the determinantal equations leads to the other parameters.

To use a sample to estimate the parameters of the model it is proposed to apply the same methods to the relative frequencies observed in a sample. A numerical example is given which shows the computations to be performed. In another paper it will be shown that these estimates have an asymptotic joint normal distribution as the sample size increases. The limiting means are the parameters estimated; the limiting variances and covariances are functions of the underlying probabilities of responses, which can be estimated consistently from the sample. These results can be used in the usual way to obtain confidence regions and tests of hypotheses.

Green (**3**) has given another method for estimating these parameters. The two methods are based on essentially the same algebraic analysis of the latent structure model. Green's method, however, uses more of the data from the sample and uses it in such a fashion that all items are treated in the same way. It would appear that in many cases Green's method or some minor modification of it (e.g., replacing P_0 by other linear combinations of P_k in Green's notation) would be more efficient than the method presented in this paper.

An advantage of the method presented here over Green's method is the simplicity of computation; in fact, these computations can be done mechanically (for example, on punched-card equipment). Green's method has the inherent difficulty that in the matrices of data to be used certain elements cannot be observed and these elements must be approximated in some way; in the present method this difficulty is avoided by treating the data asymmetrically.

Another reason for studying the present method is that the availability of large sample distribution theory makes possible obtaining standard errors of estimate, confidence intervals, and tests of hypotheses and thereby gives methods for studying the design of models for getting the best determination of the parameters. Some studies of this sort are now being made; these studies together with the analysis of some sampling experiments give suggestions for combining various estimates of the type treated here.

It seems reasonable to expect that neither the estimation method of Green nor that of this paper yields efficient estimates in the sense of estimates with minimum asymptotic variance. To obtain efficient estimates one would try the method of maximum likelihood, minimum χ^2, or another of what Neyman calls Best Asymptotic Normal estimation procedures (**4**). The methods of Green and this paper could be useful as initial approximations in an iterative method leading to efficient estimates.

2. *The Latent Class Structure*

We assume that individuals respond positively or negatively to each of a set of K items. The probability of drawing an individual from the population and obtaining from him a positive response on item j is denoted by π_j (a negative response by $\pi_{\bar{j}}$.), positive responses on items i and j by π_{ij} (a positive response on item i and a negative one on item j by $\pi_{i\bar{j}}$.), etc. The basic probabilities are the 2^K π's with K subscripts $\pi_{12\cdots K}$, $\pi_{12\cdots K-1,\bar{K}}$, etc.; the π's with less than K subscripts are linear combinations of them. Since there are K dichotomous items there are 2^K different possible response patterns. These represent 2^K mutually exclusive, and exhaustive events, and, therefore, the π's define a multinomial distribution for these 2^K categories.

In the latent class model it is assumed that the population can be divided into m (mutually exclusive) "homogeneous" subpopulations or classes. Let ν^α be the probability that a person drawn at random is drawn from the α-th subpopulation ($\alpha = 1, \cdots, m$). Let λ_i^α be the probability that an individual from the α-th subpopulation responds positively to item i ($\lambda_{\bar{i}}^\alpha = 1 - \lambda_i^\alpha$ that he responds negatively). The crucial assumption of latent structure analysis is that the probability of a collection of responses of an individual from a given class is the product of the probabilities of the responses to individual items of the collection. For example, the probability for an individual in class α of positive response to items i and j is $\lambda_i^\alpha \lambda_j^\alpha$. For an individual in a given class, therefore, the responses to the various items are statistically independent.

In this model the π's are functions of the ν's and λ's, for the probability of drawing an individual and getting a certain response pattern is the sum over the latent classes of the probability of being drawn from a class multiplied by the probability for an individual in that class of giving the response pattern. For instance

$$
\begin{aligned}
\pi_i &= \sum_\alpha \nu^\alpha \lambda_i^\alpha, && \\
\pi_{ij} &= \sum_\alpha \nu^\alpha \lambda_i^\alpha \lambda_j^\alpha, && i \neq j, \\
\pi_{ijk} &= \sum_\alpha \nu^\alpha \lambda_i^\alpha \lambda_j^\alpha \lambda_k^\alpha, && i \neq j, j \neq k, k \neq i, \\
\pi_{ij\bar{k}} &= \sum_\alpha \nu^\alpha \lambda_i^\alpha \lambda_j^\alpha \lambda_{\bar{k}}^\alpha, && i \neq j, j \neq k, k \neq i,
\end{aligned} \tag{2.1}
$$

The class to which an individual belongs cannot be observed directly; it can be inferred only from the response pattern which he gives. In order to make this inference the investigator must use the parameters of the latent structure, that is, the ν's and λ's, together with the observed response pattern of each individual.

In principle the investigator can determine directly (i.e., by means of

an infinitely large sample) the underlying probabilities of responses, the π's. Our problem is to use the π's to obtain the ν's and λ's. Essentially this means solving equations (2.1) (extended to π's with all possible subscripts). We shall give a method for doing this. The method of statistical estimation proposed here is the same but applied to estimates of the π's.

Let

$$\Lambda = \begin{bmatrix} 1 & \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_K^1 \\ 1 & \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_K^2 \\ \vdots & \vdots & \vdots & & \\ 1 & \lambda_1^m & \lambda_2^m & \cdots & \lambda_K^m \end{bmatrix} \tag{2.2}$$

be the matrix of latent parameters. The superscript refers to the row and denotes the class; the subscript refers to the column and denotes the item. It will be convenient to refer to the α-th element in the first column as λ_0^α $(=1)$ and consider it as referring to a dummy item that always has a positive response.

Let

$$N = \begin{bmatrix} \nu^1 & 0 & \cdots & 0 \\ 0 & \nu^2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \nu^m \end{bmatrix}. \tag{2.3}$$

Let Λ_1 be the $m \times m$ matrix consisting of the first m columns of Λ, and let Λ_2 be the $m \times m$ matrix consisting of the first column and the next $m - 1$ columns of Λ not contained in Λ_1 [i.e., the first row of Λ_2 is $(1\ \lambda_m^1\ \cdots\ \lambda_{2m-2}^1)$]. Let

$$\Delta = \begin{bmatrix} \lambda_k^1 & 0 & \cdots & 0 \\ 0 & \lambda_k^2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_k^m \end{bmatrix},$$

where $k > 2m - 2$. Since the initial numbering of the items is arbitrary, it is seen that we have selected from all of the items two sets of $m - 1$ items and one additional item. This implies that the battery contains at least $2m - 1$ items. (Other methods may not necessarily require so many items.)

To deduce the selected λ's and ν's we shall use the equations of (2.1) for π_{ijk} and π_{ij}, $i = 0, 1, 2, \cdots, m - 1$ and $j = 0, m, m + 1, \cdots, 2m - 2$.

When a subscript is 0, the subscript is suppressed; for example, $\pi_{0jk} = \pi_{jk}$, $\pi_{00k} = \pi_{0k} = \pi_k$. We also define $\pi_{000} = \pi_{00} = \pi_0 = 1$. Let

$$\Pi = \begin{bmatrix} \pi_{00k} & \pi_{0mk} & \pi_{0,m+1,k} & \cdots & \pi_{0,2m-2,k} \\ \pi_{10k} & \pi_{1mk} & \pi_{1,m+1,k} & \cdots & \pi_{1,2m-2,k} \\ \pi_{20k} & \pi_{2mk} & \pi_{2,m+1,k} & \cdots & \pi_{2,2m-2,k} \\ \vdots & \vdots & \vdots & & \\ \pi_{m-1,0k} & \pi_{m-1,mk} & \pi_{m-1,m+1,k} & \cdots & \pi_{m-1,2m-2,k} \end{bmatrix}$$

$$= \begin{bmatrix} \pi_{k} & \pi_{mk} & \pi_{m+1,k} & \cdots & \pi_{2m-2,k} \\ \pi_{1k} & \pi_{1mk} & \pi_{1,m+1,k} & \cdots & \pi_{1,2m-2,k} \\ \pi_{2k} & \pi_{2mk} & \pi_{2,m+1,k} & \cdots & \pi_{2,2m-2,k} \\ \vdots & \vdots & \vdots & & \\ \pi_{m-1,k} & \pi_{m-1,mk} & \pi_{m-1,m+1,k} & \cdots & \pi_{m-1,2m-2,k} \end{bmatrix}. \tag{2.4}$$

The subscript k is on each element of Π, the row subscripts (the first subscripts when not suppressed) correspond to the items in Λ_1 and the column subscripts (the second subscripts when not suppressed) correspond to the items in Λ_2. Finally let Π^* be defined the same as Π except that each element π_{ijk} is replaced by π_{ij}. The fundamental equations are

$$\Pi = \Lambda_1' \mathrm{N} \Delta \Lambda_2 , \qquad \Pi^* = \Lambda_1' \mathrm{N} \Lambda_2 . \tag{2.5}$$

These can easily be verified from (2.1).

The roots $\theta^1, \cdots, \theta^m$ of the determinantal equation

$$| \Pi - \theta \Pi^* | = 0 \tag{2.6}$$

can be determined from the probabilities of responses. This equation is

$$\begin{aligned} 0 = | \Pi - \theta\Pi^* | &= | \Lambda_1' \mathrm{N} \Delta \Lambda_2 - \theta \Lambda_1' \mathrm{N} \Lambda_2 | \\ &= | \Lambda_1' \mathrm{N} | \cdot | \Delta - \theta \mathrm{I} | \cdot | \Lambda_2 | \\ &= | \Lambda_1' | \cdot | \mathrm{N} | \cdot | \Delta - \theta \mathrm{I} | \cdot | \Lambda_2 |. \end{aligned} \tag{2.7}$$

Since Δ is a diagonal matrix, the roots of (2.7) are these diagonal elements, namely, $\lambda_k^1, \cdots, \lambda_k^m$. It should be noticed that we require Λ_1, N, Λ_2 to be nonsingular; thus every ν^α must be different from zero.

We shall now show how knowledge of the θ's together with Π and Π^* can be used to find Λ_1, Λ_2, and N.

Since $|\Pi - \theta^\alpha \Pi^*| = 0$, the matrix $(\Pi - \theta^\alpha \Pi^*)$ is singular and it is possible to find a (column) vector x^α (not all components zero) satisfying $(\Pi - \theta^\alpha \Pi^*)x^\alpha = 0$, that is,

$$\Pi x^\alpha = \theta^\alpha \Pi^* x^\alpha. \tag{2.8}$$

If the roots $\theta^1, \cdots, \theta^m$ are different, x^α is uniquely determined except for a multiplicative constant; that is, since $(\Pi - \theta^\alpha \Pi^*)$ is of rank $m - 1$, there is one linearly independent solution of (2.8). Let $X = (x^1 \cdots x^m)$ and Θ a diagonal matrix with θ^α as the α-th diagonal element. Then if we set side by side the m equations (2.8) we have

$$\Pi X = \Pi^* X \Theta. \tag{2.9}$$

Suppose that the θ's in Θ are ordered so that $\Theta = \Delta$, the diagonal matrix appearing in (2.7). Then one solution of (2.9) for X is $X = \Lambda_2^{-1}$. This is easily verified by replacing Π and Π^* in (2.9) by (2.5), Θ by Δ, and X by Λ_2^{-1}, obtaining

$$\Lambda_1' \mathrm{N} \Delta \Lambda_2 \Lambda_2^{-1} = \Lambda_1' \mathrm{N} \Lambda_2 \Lambda_2^{-1} \Delta. \tag{2.10}$$

When the θ's are ordered, X is uniquely determined except for multiplication on the right by an arbitrary (nonsingular) diagonal matrix (i.e., multiplication of each column of X by a constant). Thus any solution X can be expressed as $X = \Lambda_2^{-1} E_x$, where E is a diagonal matrix. Conversely given a solution X, we have $\Lambda_2 = E_x X^{-1}$. Since the elements of the first column of Λ_2 must be unity, each diagonal element of E_x must be the reciprocal of the first element of the corresponding row of X^{-1}. Thus the right-sided characteristic vectors of Π in terms of Π^* determine Λ_2.

Now consider the row vector $(y^\alpha)'$ satisfying $(y^\alpha)'(\Pi - \theta^\alpha \Pi^*) = 0$. Transposition gives

$$\Pi' y^\alpha = \theta^\alpha \Pi^{*\prime} y^\alpha, \tag{2.11}$$

where $\Pi' = \Lambda_2' \mathrm{N} \Delta \Lambda_1$ and $\Pi^{*\prime} = \Lambda_2' \mathrm{N} \Lambda_1$. This is the same as (2.8) with x^α replaced by y^α, and Λ_1 and Λ_2 interchanged. The argument used previously shows that $\Lambda_1 = E_y Y^{-1}$, where $Y = (y^1 \cdots y^m)$ is a set of vectors satisfying (2.11) and E_y is a diagonal matrix in which each diagonal element is the reciprocal of the first element in the corresponding row of Y^{-1}.

Finally, from (2.5) we obtain

$$(\Lambda_1')^{-1} \Pi^* \Lambda_2^{-1} = (\Lambda_1')^{-1} \Lambda_1' \mathrm{N} \Lambda_2 \Lambda_2^{-1} = \mathrm{N}. \tag{2.12}$$

We assumed earlier that the θ's were numbered to correspond to the λ_k^α. This is the only arbitrariness in our solution. It is obvious, however, from the symmetry of (2.1) that this is inherent in the model.

We can avoid the inversion of matrices by noting that

$$\Pi^*X = (\Lambda_1'\mathrm{N}\Lambda_2)(\Lambda_2^{-1}E_x) = \Lambda_1'\mathrm{N}E_x\,, \tag{2.13}$$

$$\Pi^{*\prime}Y = (\Lambda_2'\mathrm{N}\Lambda_1)(\Lambda_1^{-1}E_y) = \Lambda_2'\mathrm{N}E_y\,, \tag{2.14}$$

$$\Lambda_1 Y = \Lambda_1\Lambda_1^{-1}E_y = E_y = \mathrm{N}^{-1}(\mathrm{N}E_y). \tag{2.15}$$

Since the first row of Λ_1' consists of 1's, the elements of the first row of Π^*X are the diagonal elements of $\mathrm{N}E_x$; if we divide each column of Π^*X by its leading element we obtain Λ_1' . Similarly from $\Pi^{*\prime}Y$ we obtain $\mathrm{N}E_y$ and Λ_2' . Finally from $\Lambda_1 Y$ and $\mathrm{N}E_y$ we find N^{-1} and hence N.

3. *Estimation of Parameters*

The latent class model is to be used when the investigator has drawn a sample of dichotomous response patterns and has reason to believe that the joint probabilities have the structure indicated by equations (2.1). The investigator will estimate the π's by the corresponding p's, which are the observed relative frequencies. For example, the estimate of π_i is p_i , the proportion of respondents giving a positive response to the i-th item. Now the question is how to estimate the λ's and ν's from the p's.

It is proposed here to estimate these by applying the results of Section 2 to the p's. In this way we estimate the probabilities of the latent classes and the probabilities associated with $2m - 1$ items by treating 2 $m \times m$ matrices of p's. To obtain estimates of the probabilities associated with other items we must use other pairs of $m \times m$ matrices. We presume that the investigator knows what m is. If he does not, he can assume a value for m and compare the observed p's with up to K subscripts with the estimates of the π's computed on the estimated λ's and ν's.

It is clear that in use of this method of estimation, there is a choice of which matrices to use for estimating particular parameters. In the population it did not matter which π's were used for deriving the ν's and λ's (as long as the matrices were nonsingular) because the assumption of the model implies that the set of equations is consistent in the sense that any set of equations that can be solved will give the same ν's and λ's. In the sample, however, the values of the estimates of ν's and λ's will depend on the p's used (because the p's will not usually satisfy the conditions of algebraic consistency). This raises questions as to the choice of p's and as to how to combine different estimates of the same parameters. The asymptotic distribution theory developed elsewhere suggests answers to these questions; roughly speaking the larger the determinants of Λ_1 and Λ_2 are the better will be the estimates.

4. *An Example*

For a numerical example of the proposed method we have used some artificial data. The RAND Corporation has constructed observations from a latent structure by using random numbers in conjunction with a theoretical

structure. We give in the table below the probabilities of the latent classes and the probabilities of positive response on several items by individuals in the different latent classes. From 4000 experimental observations, four samples of 1000 each were constructed. We use one of these samples for illustrative purposes. Of course, such an example avoids many of the difficulties of working with actual data (such as determining the appropriate number of classes), but it has the obvious advantage of focussing attention on the computation.

THEORETICAL STRUCTURE

	Probability of Class	Probability of Positive Response Item 2	Item 6	Item 3	Item 7	Item 1
Class 1	.3	.9	.2	.8	.4	.9
Class 2	.5	.7	.9	.4	.8	.5
Class 3	.2	.1	.1	.3	.3	.1

We give below the observed relative frequencies of positive responses on single items, on pairs of items and on triples of items, the triples all including Item 1.

FREQUENCIES OF POSITIVE RESPONSES

	Single Items	Pairs of Items 2	6	3	7	1
Item 2	.639	——	.379	.346	.409	.403
Item 6	.544	.379	——	.221	.410	.278
Item 3	.483	.346	.221	——	.278	.313
Item 7	.586	.409	.410	.278	——	.319
Item 1	.530	.403	.278	.313	.319	——

	Item 1 and Pairs of Items 2	6	3	7
2	——	.198	.252	.242
6	.198	——	.120	.207
3	.252	.120	——	.173
7	.242	.207	.173	——

For instance 639 out of 1000 responded positively to Item 2, 346 out of 1000 responded positively to both Items 2 and 3, and 252 out of 1000 responded positively to Items 1, 2, and 3. The numbers of the items are those of the original study, but we have ordered them 2, 6, 3, 7, 1 because we shall consider the parameters of Items 2 and 6 as constituting Λ_1, Items 3 and 7, Λ_2, and Item 1, Δ. (In other words we should have renumbered the items to fit the notation of Section 2.)

For estimation we use the sample equivalents of Π and Π*, namely,

$$P = \begin{bmatrix} .530 & .313 & .319 \\ .403 & .252 & .242 \\ .278 & .120 & .207 \end{bmatrix}, \qquad P^* = \begin{bmatrix} 1.000 & .483 & .586 \\ .639 & .346 & .409 \\ .544 & .221 & .410 \end{bmatrix}. \tag{4.1}$$

First of all we find the roots of the determinantal equation

$$| P - tP^* | = \begin{vmatrix} .503 - 1.000t & .313 - .483t & .319 - .586t \\ .403 - .639t & .252 - .346t & .242 - .409t \\ .278 - .544t & .120 - .221t & .207 - .410t \end{vmatrix} = 0. \tag{4.2}$$

The determinant can be expanded to give a polynomial of third degree. The roots are $t^1 = .93661$, $t^2 = .49179$, and $t^3 = .12592$. In the matrix in (4.2) we now replace t by t^1 to give

$$P - t^1P^* = \begin{bmatrix} -.40661 & -.13938 & -.22985 \\ -.19549 & -.07207 & -.14107 \\ -.23152 & -.08699 & -.17701 \end{bmatrix}. \tag{4.3}$$

This matrix is singular. We take the set of cofactors of any row to be the elements of the estimate of the vector x^1, say $\hat{x}^1$; if we use cofactors of the first row, the components of this vector are $-.07207 \times (-.17701) - (-.08699)(-.14107) = .000{,}485{,}43$, $-.001{,}943{,}16$, and $.000{,}320{,}03$. Since a constant of proportionality is arbitrary, we multiply by 10^4. In turn we replace t by t^2 and t^3 and from each matrix use cofactors of the first row to give $\hat{x}^2$ and $\hat{x}^3$. Thus

$$\hat{X} = (\hat{x}^1\hat{x}^2\hat{x}^3) = \begin{bmatrix} 4.8543 & -\ 2.2646 & 148.2538 \\ -19.4316 & -\ 4.8783 & -102.0329 \\ 3.2003 & 14.6898 & -139.3757 \end{bmatrix}. \tag{4.4}$$

We then compute the matrix $P^*\hat{X}$. The elements in the first row of $P^*\hat{X}$ are -2.65579, 3.98740, 17.29775; these are the diagonal elements of $\hat{N}\hat{E}_x$. We divide each element of the first column of $P^*\hat{X}$ by -2.65579, etc., to obtain the estimate of Λ_1, namely,

$$L_1' = \begin{bmatrix} 1.00000 & 1.00000 & 1.00000 \\ .87074 & .72056 & .14026 \\ .12859 & .93113 & .05531 \end{bmatrix}. \tag{4.5}$$

We now use the transpose of (4.3) [that is, $(P - t^1P^*)'$] to find the first column of $\hat{Y}$ (whose elements are 4.8543, -46.7700, 30.9705), and similarly

we find the other columns of $\hat{Y}$. The first row of $P^{*\prime}\hat{Y}$ (namely, -8.18378, 24.68949, 25.61189) gives the diagonal elements of $\hat{N}\hat{E}_v$. The estimate of Λ_2' is

$$L_2' = \begin{bmatrix} 1.00000 & 1.00000 & 1.00000 \\ .85453 & .37933 & .29115 \\ .43823 & .78576 & .25307 \end{bmatrix}. \tag{4.6}$$

Next we compute

$$L_1\hat{Y}(\hat{N}\hat{E}_v)^{-1} = \begin{bmatrix} 3.89640 & .00074 & -.00248 \\ .00108 & 1.86281 & .00816 \\ -.00092 & .00195 & 4.81361 \end{bmatrix}. \tag{4.7}$$

This should be a diagonal matrix; the size of the nondiagonal elements indicates the accuracy of the computation. We take the diagonal elements to be the estimates of the diagonal elements of $\hat{N}^{-1}$. The reciprocals of the diagonal elements of (4.7) are the estimates of the diagonal elements of $\hat{N}$, namely, .25665, .53682, .20774. The estimated structure is given in the following table. This is to be compared with the theoretical structure given earlier.

ESTIMATED STRUCTURE

	Probability of Class	Probability of Positive Response				
		Item 2	Item 6	Item 3	Item 7	Item 1
Class 1	.257	.871	.129	.855	.438	.937
Class 2	.537	.721	.931	.379	.786	.492
Class 3	.208	.140	.055	.291	.253	.126

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SOLUTION OF THE PERSONNEL CLASSIFICATION PROBLEM WITH THE METHOD OF OPTIMAL REGIONS*

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The personnel classification problem is identified mathematically with other problems in the social and biological sciences. This mathematical problem is shown to be a special case of the general mathematical problem of linear programming. It is proposed here that the personnel classification problem may be solved directly by methods particularly appropriate to it as well as by the simplex method, which is a standard method for solv. g the general linear programming problem. The method of optimal regions is derived and illustrated in this paper.

1. *Introduction*

The general type of problem considered in this paper is first illustrated with a trivial problem. A business concern needs to fill three jobs which demand different abilities and training. Three applicants who can be hired for identical salaries are available. Because of the different abilities, training, and experiences, however, the value of each applicant to the company depends upon the job in which he is placed. The estimate of the value of each applicant to the company each year if he were to be assigned to any one of the three jobs is given in Table 1.

It is desired to assign the applicants to the jobs in such a way that the total value to the company is as great as possible. In this problem there are $3! = 6$ possible alternative assignments of the three men and the greatest estimated value to the company, $24,000 per year, is obtained by assigning applicant 1 to job 3, applicant 2 to job 1, and applicant 3 to job 2.

A more general problem is arrived at by denoting by c_{ij} the contribution of individual i to the common effort if he is assigned to job j, in units of the measure of the common effort. Thus in Table 1, $c_{11} = 5$, $c_{12} = 4$, $c_{13} = 7$, $c_{21} = 6$, etc. Sometimes dimensionless c_{ij} may be used to express the relative contributions to the common effort.

A more general problem features N individuals and N jobs in which, as above, the contribution of individual i to the common effort is c_{ij} units if he

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is assigned to job j. The assignments are then made so that the sum of the corresponding c_{ij} values is a maximum. We set $x_{ij} = 1$ when individual i is assigned to job j and $x_{ij} = 0$ otherwise. Then

$$\sum_i x_{ij} = 1, \qquad \sum_j x_{ij} = 1, \qquad \sum_{i,j} x_{ij} = N, \tag{1.1}$$

and the problem is to maximize*

$$T = \sum_{i,j} x_{ij}c_{ij} \, . \tag{1.2}$$

There are $N!$ values of T, though some of them may be equal, and for N large it becomes impractical to examine them all to determine the maximum. Improved methods of solution are needed.

Frequently there are many identical jobs or jobs which, though not identical, demand the same basic qualifications indicated by the c_{ij} . Such jobs can be combined into a job category. If there are m categories and if the number of jobs grouped in the jth job category is q_j , we have

$$\begin{aligned} x_{ij} &= 1 \text{ if } i \text{ is assigned to } j \\ x_{ij} &= 0 \text{ otherwise,} \end{aligned} \tag{1.3}$$

and

$$\left.\begin{aligned} \sum_{i=1}^{N} x_{ij} &= q_j \\ \sum_{j=1}^{m} x_{ij} &= 1 \end{aligned}\right\} \quad \text{with} \quad \sum_{i,j} x_{ij} = \sum_j q_j = \sum_i 1 = N. \tag{1.4}$$

We wish to maximize the total contribution to the common effort

$$T = \sum_{i,j} x_{ij}c_{ij} \, . \tag{1.5}$$

The q_j values are called quotas. This formulation of the problem is illustrated in Table 2.

Sometimes there are different individuals having identical c_{ij} values or at least the values are close enough so that the individuals may be grouped without serious error. In this case we group these individuals into personnel categories. If there are n personal categories and the number of individuals grouped in the ith personnel category is f_i , we have

$$\begin{aligned} x_{ij} &= 1 \text{ if some individual in } i \text{ is assigned to } j \\ x_{ij} &= 0 \text{ otherwise,} \end{aligned} \tag{1.6}$$

*In certain situations, particularly when the smaller values of c_{ij} indicate the greater contributions to the common effort, the values of T should be minimized.

TABLE 1

Estimated Value Per Year

in Units of $1000

		Job		
		1	2	3
Applicant	1	5	4	7
	2	6	7	3
	3	8	11	2

TABLE 2

Use of Job Categories

$i \backslash j$	1	2	3	...	j	...	m	
1	c_{11}	c_{12}	c_{13}	...	c_{1j}	...	c_{1m}	1
2	c_{21}	c_{22}	c_{23}	...	c_{2j}	...	c_{2m}	1
3	c_{31}	c_{32}	c_{33}	...	c_{3j}	...	c_{3m}	1
...	...	...	...	...	...	...	...	1
i	c_{i1}	c_{i2}	c_{i3}	...	c_{ij}	...	c_{im}	1
...	...	...	...	...	...	...	...	1
N	c_{N1}	c_{N2}	c_{N3}	...	c_{Nj}	...	c_{Nm}	1
q_j	q_1	q_2	q_3	...	q_j	...	q_m	N

TABLE 3

Use of Personnel Categories

$i \backslash j$	1	2	3	...	j	...	N	f_i
1	c_{11}	c_{12}	c_{13}	...	c_{1j}	...	c_{1N}	f_1
2	c_{21}	c_{22}	c_{23}	...	c_{2j}	...	c_{2N}	f_2
3	c_{31}	c_{32}	c_{33}	...	c_{3j}	...	c_{3N}	f_3
...	...	...	...	...	...	...	...	...
i	c_{i1}	c_{i2}	c_{i3}	...	c_{ij}	...	c_{iN}	f_i
...	...	...	...	...	...	...	...	...
n	c_{n1}	c_{n2}	c_{n3}	...	c_{nj}	...	c_{nN}	f_n
q_j	1	1	1	...	1	...	1	N

TABLE 4

Use of Personnel Categories and Job Categories

$i \backslash j$	1	2	3	...	j	...	m	f_i
1	c_{11}	c_{12}	c_{13}	...	c_{1j}	...	c_{1m}	f_1
2	c_{21}	c_{22}	c_{23}	...	c_{2j}	...	c_{2m}	f_2
3	c_{31}	c_{32}	c_{33}	...	c_{3j}	...	c_{3m}	f_3
...	...	...	...	...	...	...	...	...
i	c_{i1}	c_{i2}	c_{i3}	...	c_{ij}	...	c_{im}	f_i
...	...	...	...	...	...	...	...	...
n	c_{n1}	c_{n2}	c_{n3}	...	c_{nj}	...	c_{nm}	f_n
q_j	q_1	q_2	q_3	...	q_j	...	q_m	N

TABLE 5

Identification of Personnel Classification

with Linear Programming

General	Personnel (Generalization of Table 1)	Personnel (Table 2)	Personnel (Table 3)	Personnel (Table 4)
i	i	i	i	i
j	j	j	j	j
n	N^2	Nm	Nn	mn
λ_i	x_{ij}	x_{ij}	x_{ij}	x_{ij}
c_i	c_{ij}	c_{ij}	c_{ij}	c_{ij}
m	$2N$	$N+m$	$N+n$	$m+n$
b_j	1	q_j or 1	1 or f_i	f_i or q_j
a_{ij}	1 or 0	1 or 0	1 or 0	X or 0

and

$$\left.\begin{aligned} \sum_{i=1}^{n} x_{ij} &= 1 \\ \sum_{j=1}^{N} x_{ij} &= f_i \end{aligned}\right\} \quad \text{with} \quad \sum_{i,j} x_{ij} = \sum_i f_i = \sum_j 1 = N. \tag{1.7}$$

The values of f_i are called frequencies. This formulation of the problem is illustrated in Table 3.

A common form of the problem uses both personnel categories and job categories. If, as before, f_i is the number of persons in personnel category i and q_j is the number to be placed in job category j, then

x_{ij} = zero or a positive integer indicating the number of persons in personnel category i assigned to job category j. (1.8)

Then,

$$\left.\begin{aligned} \sum_{j}^{m} x_{ij} &= f_i \\ \sum_{i}^{n} x_{ij} &= q_j \end{aligned}\right\} \quad \text{with} \quad \sum_{i,j} x_{ij} = \sum_i f_i = \sum_j q_j = N. \tag{1.9}$$

This formulation of the problem is illustrated in Table 4.

Table 3 is a special case of Table 4 when $q_j = 1$; Table 2 is a special case of Table 4 when $f_i = 1$, and the square N by N matrix results when $f_i = 1$ and $q_j = 1$.

The problem under discussion is closely related to problems previously discussed in *Psychometrika* by Brogden (**1**), Thorndike (**9**), Votaw (**11**) and Lord (**5**). The formulation of Table 2, used essentially by Brogden (**1**, 149), Thorndike (**9**, 233), and Lord (**5**, 297), appears to be a natural form of the personnel classification problem in which several or many men are to be assigned to a relatively small number of job categories. The formulation in Table 4 has been used by Votaw (**11**, 257) in identifying the problem as a linear programming problem.

2. *Equivalent Mathematical Problems*

This problem is essentially the equivalent, mathematically, of problems arising in other applied fields. For example, it is similar to the Hitchcock transportation problem, solutions and theory of which have been supplied by many, including Dantzig (**3**) and Flood (**6**). This calls for the shipping of x_{ij} unit parcels from origin i to destination j when a_i parcels are to be shipped from origin i, b_j parcels are to be delivered to destination j, so that the total transportation cost $\sum_{i,j} x_{ij}c_{ij}$ is to be a minimum if c_{ij} is the cost of shipping a unit parcel from origin i to destination j. Equations (1.8), (1.9), and Table

4 are immediately applicable to this problem if f_i is replaced by a_i and q_j by b_j.

Rao (**7**, 322–329) has shown how different problems in biometric classification reduce to the mathematical problem discussed above. In one of these, for example, it is desired to classify an individual into one of m classes on the basis of the measurements of the individual. We desire to minimize the total risk when c_{ij} is the expected risk of erroneous classification resulting from placing individual i in class j when he does not belong there. The equations (1.6), (1.7), and Table 2 are applicable.

Also, von Neumann (**10**) has related the personnel classification problem to a certain zero-sum two-person game.

These problems and other similar ones are all special cases of the general problem of linear programming. Dantzig (2) has presented a mathematical model which is applicable to many problems, after reduction, in linear programming as follows:

Find the values of $\lambda_1, \lambda_2, \cdots, \lambda_n$ which maximize the linear form

$$\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_n c_n$$

subject to the conditions that $\lambda_j \geq 0, j = 1, 2, \cdots, n$, and

$$\begin{aligned}
\lambda_1 a_{11} + \lambda_2 a_{12} + \cdots + \lambda_n a_{1n} &= b_1 \\
\lambda_1 a_{21} + \lambda_2 a_{22} + \cdots + \lambda_n a_{2n} &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_1 a_{m1} + \lambda_2 a_{m2} + \cdots + \lambda_n a_{mn} &= b_m ,
\end{aligned}$$

where a_{ij}, b_i, c_j are constants $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

The personnel classification problem is a special case of this problem. The correspondence between the general problem and (a) the problem expressed in the individual form of which Table 1 is a special case when $N = 3$, (b) the forms of Tables 2, 3, and 4 is indicated in Table 5.

Since the classification problem, which is hereafter used to designate the mathematical problem (including minimization) encompassing the alternative problems described above, is a special case of linear programming, it follows that methods of solution of the general linear programming problem are available. In particular the simplex method is available. This has been pointed out by Votaw (**11**) to be a solution of the classification problem and furthermore a solution that in some cases can be carried out with a high speed computing machine. Even though modifications of the simplex method, as applied to the classification problem, are possible, this is not the only useful method available. It is the purpose of this paper to introduce the method of optimal regions and to show how it can be used.

It does not appear at first that the personnel classification problem is considerably simpler, mathematically, than the general linear programming problem. However, examination of the last line of Table 5 shows that the a_{ij} are integral. This fact seems to indicate a much simpler solution than is possible with the less restricted a_{ij}, for example, those which usually appear in game theory. The implication is that we should solve the classification problem directly from the mathematical model rather than to use a method of solution which is designed to handle the more general problem.

3. *Different Forms of the Problem*

Tables 1, 2, 3, and 4 illustrate four different forms in which the problem may appear. We may call the form of Table 1, which features an N by N matrix, the *individual form* since neither individuals nor jobs are grouped; the form of Table 2, the *quota form* since the quotas for each job category are specified; the form of Table 3, the *frequency form* since the frequencies of the personnel categories are specified; and the form of Table 4, the *frequency-quota form*. Actually the frequency form appears seldom in classification problems so we can drop it from further consideration and use the term "frequency form" to described the situation in Table 4. In fact, this frequency form is usually not the natural form of the classification problem but results from the grouping of individuals which is demanded when the problem is to be worked by the previously available, simplex method.

Each of these forms describes a situation in which the number of men is identical with the number of jobs. Alternative situations can be reduced formally to the forms above with the introduction of dummy men, or dummy jobs, having $c_{ij} = 0$. This is illustrated in Table 6, where the c_{ij} values for ten men and two job categories are given. In problem (a) the quotas are $q_1 = 4$ and $q_2 = 4$. It is sufficient to have a dummy job category with $q_3 = 2$ to complete the assignment. In problem (b) the quotas are $q_1 = 6$ and $q_2 = 6$. Two dummy men are introduced.

Occasionally a problem arises in which no quotas are specified. The solution in this case is extremely simple. Assign each person according to his job profile, i.e., to the job category for which his c_{ij} value is largest. This solution gives the greatest possible T, irrespective of quotas, and is sometimes included with that of the quota solution in order to show the effect of the quota restrictions.

4. *The Case of Two Job Categories*

There is no assignment problem when there is only one job category. For the case of complete quotas all individuals are assigned to this job category and, for the case of partial quotas, the individuals are assigned in the decreasing order of the c_{i1} values.

The problem really begins with two job categories. As was pointed out

TABLE 6

(a). Personnel Surplus

q_j	4	4	2
i	c_{i1}	c_{i2}	c_{i3}
1	23	13	0
2	23	17	0
3	30	20	0
4	26	22	0
5	28	25	0
6	29	27	0
7	29	30	0
8	25	33	0
9	25	37	0
10	30	49	0

(b). Personnel Deficiency

q_j	6	6
i	c_{i1}	c_{i2}
1	23	13
2	23	17
3	30	20
4	26	22
5	28	25
6	29	27
7	29	30
8	25	33
9	25	37
10	30	49
11	0	0
12	0	0

TABLE 8

Subtraction of k from c_{12}

	4	6				
	4-1	5-1				
	5-1	4-1				
	6-0	4-0				
q_j	4	6				
i	c_{i1}	c_{i2}	$J^{(1)}$	$J^{(2)}$	$J^{(3)}$	J
1	23	13	1	1	1	1
2	23	17	1	1	1	1
3	30	20	1	1	1	1
4	26	22	1	1	1	1
5	28	25	1	1	1,2	2
6	29	27	1	1,2	2	2
7	29	30	2	2	2	2
8	25	33	2	2	2	2
9	25	37	2	2	2	2
10	30	49	2	2	2	2
c_j	0	0	308			
c_j	0	-2		308		
c_j	0	-3				303

TABLE 7

Use of Differences

q_j	4	6						
i	c_{i1}	c_{i2}	$c_{i1}-c_{i2}$	J	c_{iJ}	$c_{i1}-c_{i2}-3$	J	$c_{i2}+3$
1	23	13	10	1	23	7	1	16
2	23	17	6	1	23	3	1	20
3	30	20	10	1	30	7	1	23
4	26	22	4	1	26	1	1	25
5	28	25	3	2	25	0	2	28
6	29	27	2	2	27	-1	2	30
7	29	30	-1	2	30	-4	2	33
8	25	33	-8	2	33	-11	2	36
9	25	37	-12	2	37	-15	2	40
10	30	49	-19	2	49	-22	2	52
					$\overline{\overline{303}}$			

TABLE 9

Individual Form with Optimal Regions

Units of $1000 Per Year

$j \longrightarrow$	1	2	3		
$q_j \longrightarrow$	1	1	1		
i	c_{i1}	c_{i2}	c_{i3}		J
1	5	4	7	3	3
2	6	7	3	1	1
3	8	11	2	1,2	2
$v_j \longrightarrow$	8	11	7		24

TABLE 10

Quota Form with Optimal Regions

q	4	1	4	1						
$q_j^{(4)}$	4-1	1-0	3-2	0-1						
$q_j^{(3)}$	4-1	1-0	4-1	0-0						
$q_j^{(2)}$	5-0	2-0	4-0	0-0						
$q_j^{(1)}$	6-0	0-0	4-0	0-0						
$q_j^{(0)}$	4-0	2-0	0-0	4-0						
q_j	4	1	4	1	$J^{(0)}$	$J^{(1)}$	$J^{(2)}$	$J^{(3)}$	$J^{(4)}$	J
10	23	13	16	14	1	1	1	1	1	1
20	23	17	27	37	4	3	3	3	3	3
30	30	20	20	16	1	1	1	1	1	1
40	26	22	22	15	1	1	1	1,3	1,3	3
50	28	25	30	41	4	3	3	3	3,4	4
60	29	27	20	13	1	1	1	1	1	1
70	29	30	36	38	4	3	3	3	3	3
80	25	33	20	22	2	1	1	1	1	1
90	25	37	33	38	4	3	3	3	3	3
100	30	49	25	26	2	1	2	2	2	2
$v_j^{(0)}$	0	0	0	0	344					
$v_j^{(1)}$	29	49	27	41		289				
$v_j^{(2)}$	30	48	27	40			308			
$v_j^{(3)}$	31	48	27	39				308		
$v_j^{(4)}$	31	48	27	38						315

by Thorndike (**9**) and Lord (**5**), however, this problem, for complete quotas, is easily solved without extensive theory by the use of subtraction.

Consider the c_{ij} values of Table 7 with $q_1 = 4$ and $q_2 = 6$. It is only necessary to form the difference $c_{i1} - c_{i2}$ for all i and to assign the individuals to job category 1 in the order of the largest value of this difference and to job category 2 in the order of the smallest value of this difference. The method is illustrated in Table 7.

The symbol J is used to indicate the assignment so that $x_{iJ} = 1$ and any other $x_{ij} = 0$. The value of c_{iJ} is the value of c_{ij} corresponding to x_{iJ} and we have

$$T = \sum_{i,j} x_{ij}c_{ij} = \sum_{i} x_{iJ}c_{iJ} . \tag{4.1}$$

In Table 7 this is 303 units where $q_1 = 4$ and $q_2 = 6$.

The sum which results when assignment is made to job category 1 when $c_{i1} - c_{i2} > 0$, to job category 2 when $c_{i2} - c_{i1} > 0$, and to either job category 1 or job category 2 when $c_{i1} - c_{i2} = 0$, is certainly an optimal one though the desired quotas are not necessarily met. Now since the assignments depend on the differences $c_{i1} - c_{i2}$, we can add any constant to $c_{i1} - c_{i2}$ without changing the resulting assignments. The value, -3, is added to each of the values of $c_{i1} - c_{i2}$ to form $c_{i1} - c_{i2} - 3$ in Table 7 so that the positive values correspond to assignments to job category 1, the negative values (including 0) to job category 2, and the quotas are met by assigning the men in accordance with the sign of $c_{i1} - c_{i2} - 3$.

The solutions for different quotas can be obtained by forming, for different values of k,

$$c_{i1} - c_{i2} + k = (c_{i1} - 0) - (c_{i2} - k). \tag{4.2}$$

Assignment is then made

$$\begin{aligned} &\text{to the first job category when} && c_{i1} - 0 > c_{i2} - k, \\ &\text{to the second job category when} && c_{i1} - 0 < c_{i2} - k, \\ &\text{to either job category when} && c_{i1} - 0 = c_{i2} - k. \end{aligned} \tag{4.3}$$

The problem of Table 7 is worked in Table 8 by subtracting different values of k. The values so subtracted are placed at the bottom of the c_{i2} column. Assignments are made using (4.3). The resulting assignments are summarized at the top of the table. The first of each pair of entries indicates the number of unique assignments and the second the number of ties. On the basis of these results a new k is selected until the quotas are met. The value of T is placed in the lower right-hand corner at each step.

If the values of c_{i1} are plotted as x's and the values of c_{i2} plotted as y's, the line $x - y + k = 0$ divides the space into "optimal regions" with the

number of points in each indicating the resulting quotas. We move the line parallel to its original position till the desired quotas are met. These optimal regions for specified quotas correspond in the m-dimensional case to the regions bounded by hyperplanes discussed by Brogden (1) and Lord (5).

5. *The Conditions of Solution**

For the mathematical problem of classification we need conditions of solution which are generalizations of (4.3). To expedite the analytic solution, y_{ij}^2 is used in place of $x_{ij} \geq 0$ in (1.8) so that the expressions (1.9) become

$$\left.\begin{aligned} \sum_j^m y_{ij}^2 &= f_i \\ \sum_i^n y_{ij}^2 &= q_j \end{aligned}\right\} \quad \text{with} \quad \sum_{ij} y_{ij}^2 = \sum_i f_i = \sum_j q_j = N. \tag{5.1}$$

We desire to maximize

$$T = \sum_{ij} y_{ij}^2 c_{ij}, \tag{5.2}$$

where y_{ij}^2 is 0 or a positive integer.

An analytic derivation of the condition of solution using differential calculus can be obtained from an extension of the problem in which y_{ij}^2 is any real number defined by

$$0 \leq y_{ij}^2 \leq N. \tag{5.3}$$

Since c_{ij} is also finite, T is bounded and there is at least one set of y_{ij}^2 resulting in the largest T. If these y_{ij}^2 are integers they constitute a solution of the desired problem. We use Lagrange multipliers for the side conditions (5.1) and have

$$\phi = \sum_{ij} c_{ij} y_{ij}^2 + \sum_i u_i(f_i - \sum_j y_{ij}^2) + \sum_j v_j(q_j - \sum_i y_{ij}^2). \tag{5.4}$$

Taking partial derivatives with respect to each y_{ij} we find

$$\begin{aligned} \frac{\partial \phi}{\partial y_{ij}} &= 2(c_{ij} - u_i - v_j) y_{ij} \\ \frac{\partial^2 \phi}{\partial y_{ij}^2} &= 2(c_{ij} - u_i - v_j) \\ \frac{\partial^2 \phi}{\partial y_{ij}\, \partial y_{i'j'}} &= 0, \qquad i \neq i', j \neq j'. \end{aligned} \tag{5.5}$$

*The initial steps in the development of this section follow the outline of a proof presented by Professor D. F. Votaw, Jr., at a conference on personnel classification problems held at Personnel Research Branch, Department of the Army, Washington, D. C. in August, 1952.

Hence the necessary conditions for a maximum sum, where the maximum sum need not be greater than any other sum but must be at least as large, are

$$\left.\begin{aligned} (c_{ij} - u_i - v_j)y_{ij} &= 0 \\ (c_{ij} - u_i - v_j) &\leq 0. \end{aligned}\right\} \tag{5.6}$$

For assigned values, $x_{ij} > 0$, $y_{ij} > 0$ and the necessary conditions are

$$c_{ij} - u_i - v_j = 0, \tag{5.7}$$

and in all other cases, $x_{ij} = 0$, $y_{ij} = 0$ and the necessary conditions are

$$c_{ij} - u_i - v_j \leq 0. \tag{5.8}$$

Since there is always a maximum sum in the sense defined above (for the complete assignment problem) there must be at least one set of u_i and v_j satisfying (5.7) and (5.8). Hence when the sum is maximal we have

$$\begin{cases} c_{ij} = u_i + v_j & \text{for assigned values} \\ c_{ij} \leq u_i + v_j & \text{for unassigned values.} \end{cases} \tag{5.9}$$

These conditions are used as the conditions of solution of the simplex method (**3**, 369) (**11**, 259). The conditions themselves are independent of the x_{ij} which are positive integers (including zero) indicating the number of assignments.

These conditions of solution are illustrated by application to the trivial problem of Table 1 in which $f_i = 1$, $q_j = 1$ so $x_{ij} = 0$ or 1. The values $v_1 = 0$, $v_2 = 2$, $v_3 = 1$, $u_1 = 6$, $u_2 = 6$, $u_3 = 9$ satisfy the first equation of (5.9) where the assignments are the 13, the 21, and the 32 terms. The other values of $c_{ij} - u_i - v_j$ are all negative so the solution is a maximum.

If we denote the assigned values by J, and the others by j, $x_{iJ} > 0$, $x_{ij} = 0$ and

$$\begin{cases} c_{iJ} - u_i - v_J = 0 \\ c_{ij} - u_i - v_j \leq 0 \end{cases} \tag{5.10}$$

are the conditions of solution.

6. *A Generalization of Brogden's Condition*

It seems preferable to eliminate the values of the u_i from (5.10) by subtraction and to write the conditions of solution as

$$c_{iJ} - c_{ij} \geq v_J - v_j \tag{6.1}$$

or the equivalent

$$c_{iJ} - v_J \geq c_{ij} - v_j \,. \tag{6.2}$$

These inequalities are the conditions of optimal solution on which the method of optimal regions is based. For the case of complete quotas there are always

maximal sums in the sense defined above and in this case the condition (6.2) holds. Hence there must be values of v_i (and v_J) for expressing this condition in any numerical problem. This is consistent with the formulation of the problem as given by Lord (**5**, 300), which calls for thed etermination of $a_i = v_i$. These values of v_i are expressible in terms of the number system in which the c_{ij} are expressed.

Now (6.1), and hence (6.2), is in a sense a generalization of a condition used by Brogden in connection with his method of differences of critical rejection scores. He suggested that (**1**, 151) "Assignment is made to the first of two assignments when the value of the difference variable exceeds that of the difference score." In the present notation we might write Brogden's condition as

$$c_{iJ} - c_{ij} \geq r_J - r_j \, , \tag{6.3}$$

where r_J and r_j are rejection scores with no assignment being made when $c_{ij} - r_j$ are all negative for all j and fixed i. The condition (6.1), or its equivalent (6.2), might well be called Brogden's generalized condition.

Rao (**7**, 344) has proved a lemma which is very similar to (6.2). He has worked out an ingenious graphical-mechanical solution of the classification problem when $m = 3$ (**7**, 327–329).

7. *The Method of Optimal Regions*

We are now in a position to apply the method of optimal regions, which consists in taking values of v_i, finding the corresponding quotas, and changing the v_i until the quotas are met. The method is illustrated first in Table 9 with the trivial problem of Table 1.

Since one man is to be selected from each column we take the initial v_j to be the highest c_{ij} value in that column. Assignments using (6.2) show that, when the tie situation resulting from its application to the c_{ij} values of man No. 3 is resolved in favor of assignment to the second job, the quotas are satisfied and the greatest sum is \$24,000 per year.

For some purposes we wish to know what is the maximum sum if there are no quota restrictions. In Table 9, for example, this amount is \$25,000 per year, which is obtained by adding the largest value in each row. The quota restrictions do not result here in a large percentage loss.

Assignments made with (6.2) with $v_j = 0$ give this completely maximal assignment. In the following illustrations a preliminary step to the method of optimal regions features the use of $v_j = 0$. The first step then features the values of v_j which are estimates of the v_j leading to the desired quotas.

The method is applied in Table 10 to a problem in the quota form. This is an abbreviation of a problem used by Brogden (**1**, 149). Every tenth man in his list was taken and the values of c_{ij} were obtained from his values by multiplying by 10 and adding 25. The quotas adopted for this problem are

$q_1 = 4$, $q_2 = 1$, $q_3 = 4$, $q_4 = 1$. The greatest possible T, without regard to quotas, is 344 units.

The initial values of v_j, the $v_j^{(1)}$, are determined by selecting the fourth largest c_{i1}, the largest c_{i2}, the fourth largest c_{i3}, and the largest c_{i4}. The tentative assignments are made using (6.2) and the resulting quotas are indicated in rows above the table. The first entry is the number of unique assignments and the second is the number of ties.

The resulting $q_j^{(1)}$ are compared with the desired q_j. The initial assignment gives an excess of 2 in the first job category, a deficiency of 1 in the second, and a deficiency of 1 in the fourth. An excess indicates the value of v_j is too low and a deficiency indicates the value of v_j is too high. The value of $v_1^{(1)}$ is raised by one unit to form $v_1^{(2)}$ and the values of $v_2^{(1)}$ and $v_4^{(1)}$ are lowered by one unit to form $v_2^{(2)}$ and $v_4^{(2)}$. The values of $J^{(2)}$ and $q_j^{(2)}$ are computed. Values of $v_j^{(3)}$ are then assigned. After four steps with the assignment of ties, the quotas are met.

The method is called the method of optimal regions, since the changing of the values of v_j may be interpreted as the moving of hyperplanes parallel to original positions in such a way that the optimal solutions involving the number of points within the resulting regions (or on the boundaries) eventually satisfy the desired quotas.

The method of optimal regions is based on the fact that bounded values of v_j exist which identify the regions associated with the desired quotas. A primitive form of the method calls for the calculation of the quotas of each different set of v_j values (or perhaps the calculation of those sets close to an approximate set such as $v_j^{(1)}$). This form of the method is not practical except in trivial problems. A less formal variation of the method, suitable for problems of hand calculation but not of machine calculation, calls for estimates of the values of the increments to the $v_j^{(k)}$ in accordance with the values of $q_j^{(k)} - q_j$. There is no guarantee, of course, unless an appropriate estimation scheme is specified, that such estimates will lead to an improvement at any specific step, but in many hand computation problems the alert computer can arrive at the result in relatively few steps. Larger numbers of job categories, larger numbers of rows, and large positive correlations between the columns of c_{ij} values tend to increase the number of steps.

A more formal variation of the method, and one which is better adapted to machine computation, calls for the calculation, and ordering for fixed j and k, of the values of $c_{ij} - v_j^{(k)} - c_{iJ} + v_J^{(k)}$. These values are all non-positive as indicated by (6.2). This detailed method of optimal regions calls for the determination of the increments to the $v_j^{(k)}$ by selecting $q_j - q_j^{(k)}$ largest values of $c_{ij} - v_j^{(k)} - c_{iJ} + v_J^{(k)}$ in the same way that the values of $v_j^{(1)}$ are selected using the largest q_j values of c_{ij}. There is no absolute guarantee that the process will converge to a solution, though the method of exhaustion does guarantee a solution after a finite number of steps. The method is simply

an iterative process in which, at each step, we see where we are and where we want to go. We take steps, which for the most part can be mechanized, toward the objective.

In practice there seems to be little difficulty in obtaining a solution in a relatively small number of steps except when the objective is not clear because of large numbers of ties. These make it difficult to discover if the q_j can be obtained from the $q_j^{(k)}$. This is precisely the problem faced by Smith (**8**, 45) with the method of bounding sets. A solution has been given by Votaw and Dailey (**12**, 16) which is practical, for hand computation, when the number of job categories is less than eight, and for larger values with machine computation. Variations in the method of interchange (**4**) may also be used in solving this problem, which has been called "the quota problem" by Votaw and Dailey (**12**, 15).

Because of the multiplicity of ties in rows having high frequencies, the quota problem is commonly more serious in connection with problems naturally expressed in the frequency form (transportation problems) than those naturally expressed in the quota form (personnel classification problems). As an illustration involving multiple ties the method of optimal regions is applied to a problem naturally expressed in the frequency form in Table 11. This is essentially a problem used by Votaw (**3**, 259) to illustrate the simplex method. The form of Table 11 presents both the statement of the problem and its solution. The values of $v_j^{(1)}$ are determined by finding the 35th of the c_{i1} values, the 35th of the c_{i2} values, and the 30th of the c_{i3} values, considering the frequencies and measuring from the largest values of c_{ij} .

Ties, many of them, result in the first and fourth man categories. Can these ties be resolved so as to satisfy the quotas? A necessary condition is

TABLE 11

Frequency Form with Optimal Regions

q_j	35	35	30						
$q_j^{(1)}$	20-60	20-20	0-40						
$q_j^{(0)}$	40-40	0-20	0-60						
q_j	35	35	30	f_i	$J^{(0)}$	$J^{(1)}$	x_{iJ}		
1	9	2	9	40	1,3	1,3	10		30
2	1	8	8	20	2,3	2		20	
3	7	2	1	20	1	1	20		
4	9	8	0	20	1	1,2	5	15	
$v_j^{(0)}$	0	0	0		840		(35)	(35)	(30)
$v_j^{(1)}$	9	8	9						825

TABLE 12

Quota Form with Rejection Scores

q_j	2	1	2	1				
$q_j^{(2)}$	3-0	0-0	2-0	1-0				
$q_j^{(1)}$	1-1	0-2	2-0	1-0				
$q_j^{(0)}$	4-0	2-0	0-0	4-0				
q_j	2	1	2	1				
10	23	13	16	14	1	-	-	-
20	23	17	27	37	4	-	-	-
30	30	20	20	16	1	1	1	1
40	26	22	22	15	1	-	-	-
50	28	25	30	41	4	4	4	4
60	29	27	20	13	1	-	1	1
70	29	30	36	38	4	3	3	3
80	25	33	20	22	2	-	-	-
90	25	37	33	38	4	3	3	3
100	30	49	25	26	2	1,2	1	2
$r_j^{(0)}$	0	0	0	0	236			
$r_j^{(1)}$	30	49	33	41		189		
$r_j^{(2)}$	29	49	33	41			199	
$r_j^{(3)}$	29	47	33	41				218

that the available values of $q_j^{(1)}$, including ties, must be greater than q_i for each value of j. We see at once from the upper rows that

$$g_1 = 20 + 60 \geq q_1 = 35; \qquad g_2 = 20 + 20 \geq q_2 = 35;$$

$$g_3 = 0 + 40 \geq q_3 = 30.$$

If this necessary condition were not met, we would take values of $v_j^{(2)}$ in the direction of meeting it. Here we apply the additional necessary conditions involving combinations of job categories:

$$g_{12} = 80 \geq q_{12} = 70; \qquad g_{13} = 80 \geq q_{13} = 65; \qquad g_{23} = 80 \geq q_{23} = 65;$$

$$g_{123} = 100 \geq q_{123} = 100.$$

The satisfaction of all these necessary conditions constitutes a sufficient condition and the quotas can be met. The resulting values of x_{ij} are shown at the right of Table 11. The value of T is 825 units and the quota restrictions cause a loss of but 15 units.

Frequently, as in this case, the method of optimal regions gives a very satisfactory solution of problems naturally expressed in the frequency form, even though the matter of ties sometimes demands close examination. The method is recommended, however, primarily for problems which are naturally expressed in a quota form featuring a relatively small m.

The method is applicable to non-trivial problems. A classification problem with 1152 man and 7 job categories, and highly correlated columns of c_{ij}, was worked with it in eight iterations.

8. *The Case of Incomplete Quotas*

A problem involving incomplete quotas can be worked with the method of optimal regions after the problem has been transformed to a problem with complete quotas by the use of the device illustrated in Table 6. However, the method of critical rejection scores may be used instead. This method is especially effective when the quotas are (relatively) small numbers. The method of rejection scores is a slight modification of the method of differences of critical rejection scores introduced by Brogden (1). This modification uses a method of solution very similar in form to that of the method of optimal regions described above.

Brogden used rejection scores in indicating a preliminary assignment. If r_j is a rejection score, assignments are tentatively made to job category j only if

$$c_{ij} - r_j \geq 0. \tag{8.1}$$

The use of (8.1) may lead to the tentative assignment to more than one job category. Conflicting assignments are then resolved by assigning the individual

to job category J in accordance with Brogden's condition (6.3). This may be written in the alternative form

$$c_{iJ} - r_J \geq c_{ij} - r_j \ . \tag{8.2}$$

It is not necessary to use (8.1) and then Brogden's condition. The same result can be obtained with the use of (8.2) and the additional specification $c_{iJ} - r_J \geq 0$. If this is not satisfied for every value $j = J$, no assignment is made.

The technique of the method follows that of the method of optimal regions with r_J playing the role of v_J, r_j of v_j, except that no assignment is made to individual i when every $c_{ij} - r_j < 0$.

The method is illustrated in Table 12 where the problem of Table 10 with $q_1 = 2$, $q_2 = 1$, $q_3 = 2$, $q_4 = 1$ is solved with the method of rejection scores. The values $r_j = 0$ are first used. The six largest c_{iJ} resulting are added to obtain the maximum sum disregarding quotas.

The method of rejection scores can be used to solve the complete assignment problem. In this case one takes successively smaller values of the r_j until the quotas are met.

9. *Concluding Remarks*

The personnel classification problem, which is mathematically the equivalent of several other problems in the social and biological sciences, is really a special problem in linear programming. Though the techniques, such as the simplex method, of the general linear programming problem are available. it seems preferable to direct the mathematical solution toward the simpler mathematical model of the classification problem. When this is done a condition for assignment, which is a generalization of Brogden's condition, is available and serves as the foundation of the method of optimal regions. This method, which is based on concepts related to those introduced by personnel workers such as Brogden, Thorndike, and Lord, seems particularly effective when applied to the personnel classification problem.

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Addition at proof reading: The detailed method of optimal regions has been further improved with the introduction of devices for automatic determination of the successive sets of u_i which converge to a solution. Preliminary transformations lead to initial values of v_j which, in all the problems worked thus far, become solutions with two or three iterations. For example, the 1152 $\times$ 7 problem mentioned above has been solved by determining sets of v_j from a preliminary transformation followed by two steps of the detailed method of optimal regions.

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THE MATHEMATICAL THEORY OF FACTORIAL INVARIANCE UNDER SELECTION

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It is first demonstrated that Aitken's selection formulas are equivalent to a linear transformation in the factor space. On this basis the Thomson-Ledermann theorem concerning the invariance of the number of common factors under selection, and a theorem concerning the invariance of factor loadings under selection are derived. A mathematical proof of the results of Thurstone, which are concerned with the invariance of simple structure under selection, is given. The paper provides a conclusive answer to the question, considered by Thurstone and Thomson, whether a multivariate selection is always reducible to successive univariate selections.

I. *Selection as a Linear Transformation*

The effect of a selection of population (i.e., selection of subjects in psychological experiments) on correlations and variances is indicated by Aitken's formulas

$$\begin{aligned} V_{ik} &= V_{ij}R_{ji}^{-1}R_{ik} = V'_{ki} \\ V_{kk} &= R_{kk} - R_{ki}(R_{ii}^{-1} - R_{ii}^{-1}V_{ii}R_{ii}^{-1})R_{ik} \end{aligned} \tag{1}$$

(see 1, 2, 3 or 4), involving the inverse R_{ii}^{-1}. As is well known, the inverse of a square matrix exists only if the matrix is nonsingular, i.e., if its determinant does not vanish. In virtue of this, we must have $| R_{ii} | \neq 0$. On the other hand, it follows from this condition that the rows and the columns of R_{ii} must be linearly independent. Accordingly, we can state the following theorem:

SELECTION THEOREM I: *Selection tests are linearly independent in a selection determined by Aitken's formulas.*

We then pass over to a geometrical mode of representation customary in factor analysis and let each test be represented by a vector of unit length. We call the manifold spanned by the l "selection test vectors" (i.e., vectors that geometrically represent the selection tests) the "selection space." Since, in virtue of Selection Theorem I, the l selection test vectors are linearly independent, the selection space is l-dimensional. It follows directly from the definition of the selection space that selection test vectors are entirely in this space. Other test vectors, on the other hand, may have—in addition to a selection space component—another component outside of this space. We then set up an arbitrary coordinate system in the selection space and express

6

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the projections of the selection test vectors on the coordinate axes by the matrix $\| a_{jm} \| = A_j$, where the representative element a_{jm} expresses the projection of jth selection test vector on the mth coordinate axis. On the other hand, let the matrix $\| a_{km} \| = A_k$ express the projections of the non-selection test vectors on the coordinate axes of the selection space, the representative element a_{km} expressing the projection of the kth non-selection test vector on the mth coordinate axis. The scalar products of selection test vectors and the selection space components of non-selection test vectors then have the form

$$
\begin{aligned}
\rho_{jJ} &= \sum_m a_{jm} a_{Jm} \,, \\
\rho_{jk} &= \sum_m a_{jm} a_{km} \,, \\
\rho_{kK} &= \sum_m a_{km} a_{Km} \,,
\end{aligned}
\tag{2}
$$

where j and J refer to selection tests and k and K to non-selection tests. As the selection test vectors are entirely in the selection space, their intercorrelations are simply

$$
\begin{aligned}
r_{jJ} &= \rho_{jJ} \,, \\
r_{jk} &= \rho_{jk} \,.
\end{aligned}
$$

On the other hand, as the non-selection test vectors are not necessarily entirely in the selection space, we have

$$
r_{kK} = \rho_{kK} + \bar{\rho}_{kK} \,,
$$

where $\bar{\rho}_{kK}$ is that part of the scalar product of the test vectors k and K which originates outside of the selection space. In matrix notation, we can express the foregoing as

$$
R_{jj} = A_j A_j' \,, \tag{3}
$$

$$
R_{jk} = A_j A_k' \,, \tag{4}
$$

$$
R_{kk} = A_k A_k' + \bar{R}_{kk} \,. \tag{5}
$$

In the selection space we then perform a linear transformation, converting the selection test vectors j into new vectors whose projections on the coordinate axes are b_{jp} . Transformation coefficients l_{mp} , l^2 in number, are determined by l^2 linear equations,

$$
b_{jp} = \sum_m a_{jm} l_{mp} \,. \qquad
\begin{matrix}
(j = 1, 2, \cdots, l) \\
(p = 1, 2, \cdots, l)
\end{matrix}
$$

The new coordinates of the non-selection test vectors can then be determined from equations

the parameters of the operator [illegible] and the amplitude [illegible] respectively [illegible] [illegible]

[illegible]

they have the form

$$A_i = \sum [illegible]$$

$$E_{ij} = \sum [illegible]$$

where ξ and [illegible]

[illegible]

On the other hand, [illegible]

$$E_{ij} = [illegible]$$

[illegible]

$$K_{ij} = A_i A_j$$

$$K_{ij} = A_i A_j$$

$$K_{ij} = A_i A_j + E_{ij}$$

In the [illegible]

$$E_{ij} = \sum [illegible]$$

The new coefficients of the [illegible] form equations.

$$b_{kp} = \sum_{m} a_{km} l_{mp} \,. \qquad \begin{matrix}(k = 1, 2, \cdots, l)\\(p = 1, 2, \cdots, l)\end{matrix}$$

The above sets of equations can be written in matrix notation as

$$B_i = A_i L, \tag{6}$$

$$B_k = A_k L, \tag{7}$$

the matrix of transformation being

$$L = A_i^{-1} B_i \,. \tag{8}$$

The scalar products of the new, transformed test vectors in the selection space are

$$\mu_{iJ} = \sum_p b_{ip} b_{Jp} \,,$$

$$\mu_{ik} = \sum_p b_{ip} b_{kp} \,,$$

$$\mu_{kK} = \sum_p b_{kp} b_{Kp} \,,$$

and the new covariances of the tests accordingly

$$v_{iJ} = \mu_{iJ} \,,$$

$$v_{ik} = \mu_{ik} \,,$$

$$v_{kK} = \mu_{kK} + \bar{\rho}_{kK} \,.$$

In matrix notation:

$$V_{ii} = B_i B_i' \,, \tag{9}$$

$$V_{ik} = B_i B_k' \,, \tag{10}$$

$$V_{kk} = B_k B_k' + R_{kk} \,. \tag{11}$$

From (10), together with (7), (8), (4), (9), and (3), we have successively

$$\begin{aligned} V_{ik} &= B_i L' A_k' = B_i B_i' (A_i')^{-1} A_i^{-1} R_{ik} \\ &= V_{ii} (A_i A_i')^{-1} R_{ik} = V_{ii} R_{ii}^{-1} R_{ik} \,, \end{aligned} \tag{12}$$

and from (11), together with (7), (5), (4), and (3), we obtain

$$\begin{aligned} V_{kk} &= A_k L L' A_k' + R_{kk} - A_k A_k' \\ &= R_{kk} - R_{ki} (A_i^{-1})' A_i^{-1} R_{ik} + R_{ki} (A_i^{-1})' L L' A_i^{-1} R_{ik} \\ &= R_{kk} - R_{ki} (A_i A_i')^{-1} R_{ik} + R_{ki} (A_i^{-1})' A_i^{-1} B_i B_i' (A_i^{-1})' A_i^{-1} R_{ik} \\ &= R_{kk} - R_{ki} R_{ii}^{-1} R_{ik} + R_{ki} R_{ii}^{-1} V_{ii} R_{ii}^{-1} R_{ik} \\ &= R_{kk} - R_{ki} (R_{ii}^{-1} - R_{ii}^{-1} V_{ii} R_{ii}^{-1}) R_{ik} \,. \end{aligned} \tag{13}$$

In point of fact, however, (12) and (13) are simply Aitken's formulas (1), so that we have proved the following:

SELECTION THEOREM II: *Selection, as regards its effects on correlations and variances, is equivalent to a linear transformation in the selection space.*

As far as a univariate selection is concerned, the matrix of transformation, L, is reduced to a single element, viz., to the new standard deviation of the single selection test. Therefore, univariate selection simply corresponds to a "contraction" or "dilatation" of the test vector space in the direction of the selection test vector (see 1, 444). The question of whether a multivariate selection can always be reduced to successive univariate selections has been considered by Thurstone and Thomson (see 1 and 2). This question is answered by our theory in the affirmative. For a linear transformation in an l-dimensional space is determined by l^2 transformation coefficients, and it is always reducible to l successive "contractions" and "dilatations" taking place in various directions of this space. A single contraction or dilatation of this kind in a given direction is uniquely determined by $l - 1$ ratios of the direction cosines together with one contraction or dilatation coefficient; $l - 1 + 1 = l$ conditions can accordingly be imposed upon any univariate selection, and consequently l^2 conditions upon l successive univariate selections. On the other hand, these l successive univariate selections obviously define a given linear transformation in the selection space. By letting this linear transformation satisfy the l^2 conditions that its l^2 transformation coefficients are l_{mp} , i.e., the same as those of the linear transformation corresponding to a given multivariate selection, we can reduce a given multivariate selection into l successive univariate selections.

SELECTION THEOREM III: *A given l-fold multivariate selection is always reducible to l successive univariate selections.*

II. *On the Invariances in Selection*

We now substitute communalities for the unities in the principal diagonals of R_{ii} and R_{kk} to obtain the reduced correlation matrices R_{ii}^* and R_{kk}^* . For the sake of uniformity we also shall subsequently denote the matrices R_{ik} and R_{ki} by R_{ik}^* and R_{ki}^* , respectively. Instead of the complete correlation matrix R we now have the reduced correlation matrix R^*. Let us assume that by factoring this matrix we obtain r orthogonal common factors. We partition the resulting common factor matrix F of order $n \times r$ into two submatrices, viz., into an $l \times r$ submatrix F_i , comprising the selection tests, and an $(n - l) \times r$ submatrix F_k , comprising the non-selection tests (see Table 1). In virtue of the basic theorem of factor analysis we then obtain

$$
\begin{aligned}
R^* &= FF', \\
R^*_{ll} &= F_l F_l', \\
R^*_{lk} &= F_l F_k' = R'_{kl}, \\
R^*_{kk} &= F_k F_k'. \qquad (14)
\end{aligned}
$$

In Table 1 we write, besides the common-factor matrices F_l and F_k, the unique-factor matrix of the selection tests, U_l, which is, as a matter of course, a diagonal matrix, and a matrix of the non-selection tests in terms of the unique factors of the selection tests, which is, as a matter of course, a null matrix. We denote the total matrix thus constructed F_0.

TABLE 1

$$
\begin{array}{c|c|c|}
 & r & l \\
\hline
l & F_l & U_l \\
\hline
n-l & F_k & O \\
\hline
\end{array}
$$

TABLE 2

$$
\begin{array}{c|c|}
 & r \\
\hline
l & F_l \\
\hline
n-l & F_k \\
\hline
r & T \\
\hline
\end{array}
$$

TABLE 3

$$
\begin{array}{c|c|c|}
 & r & l \\
\hline
l & F_l & U_l \\
\hline
n-l & F_k & O \\
\hline
\end{array}
\;\cdot\;
\begin{array}{c|c|}
 & r+l \\
\hline
r & L_c \\
\hline
l & L_u \\
\hline
\end{array}
\;=\;
\begin{array}{c|c|}
 & r+l \\
\hline
l & F_l L_c + U_l L_u \\
\hline
n-l & F_k L_c \\
\hline
\end{array}
$$

It was shown in the first section of this paper that selection is equivalent to a linear transformation in the selection space spanned by the selection test vectors. Since this l-dimensional selection space is evidently a subspace of the $(r + l)$-dimensional space comprising the r-dimensional common-factor space and the l-dimensional unique-factor space of the selection tests, a linear transformation, L, which we perform in the selection space, at the same time means a given linear transformation L_0 in the above-mentioned total space. For this reason the factor matrix F_0 is transformed in selection into the matrix F_0L_0 (see Table 2), giving the new after-selection covariances and common-factor variances according to the formula

$$
V^* = (F_0L_0)(F_0L_0)'. \qquad (15)
$$

We partition the matrix L_0 of order $(r + l) \times (r + l)$ in the manner indicated by Table 2 to get an $r \times (r + l)$ submatrix L_C, corresponding to the transformation of the common-factor space, and an $l \times (r + l)$ submatrix L_U corresponding to the transformation of the unique-factor space of the selection tests. Employing this notation, $F_i L_C + U_i L_U$ is that submatrix of $F_0 L_0$ which includes the selection tests, and $F_k L_C$ that submatrix of $F_0 L_0$ which comprises the other tests (see Table 2). Equation (15) then yields

$$\begin{aligned} V_{ii}^* &= (F_i L_C + U_i L_U)(F_i L_C + U_i L_U)', \\ V_{ik}^* &= (F_i L_C + U_i L_U)(F_k L_C)' = V_{ki}', \\ V_{kk}^* &= (F_k L_C)(F_k L_C)'. \end{aligned} \qquad (16)$$

The new, after-selection correlations and communalities are obtainable from the new, after-selection covariances and common-factor variances, represented by (15) and (16), in accordance with the formulas

$$\begin{aligned} {}_sR^* &= DV^*D = (DF_0L_0)(DF_0L_0)', \\ {}_sR_{ii}^* &= D_i V_{ii}^* D_i = (D_i F_i L_C + D_i U_i L_U)(D_i F_i L_C + D_i U_i L_U)', \\ {}_sR_{ik}^* &= D_i V_{ik}^* D_k = (D_i F_i L_C)(D_k F_k L_C)' + (D_i U_i L_U)(D_k F_k L_C)', \\ {}_sR_{kk}^* &= D_k V_{kk}^* D_k = (D_k F_k L_C)(D_k F_k L_C)', \end{aligned} \qquad (17)$$

where D, D_i, and D_k are diagonal matrices, the reciprocal values of the new, after-selection standard deviations of the tests being their elements. We see that the new, after-selection common-factor matrix is *not* DFL_C, which is obtained from the matrices $D_i F_i L_C$ and $D_k F_k L_C$, but the new common-factor matrix is DF_0L_0. The rank of DF_0L_0 accordingly indicates the number of common factors which are obtained after selection. Let us determine this rank.

We know, for one thing, that the rank of a product matrix cannot be higher than that of any factor. Consequently, the rank of DF_0L_0 is at most equal to the rank of F_0. On the other hand, the diagonal matrix D always has an inverse, and the square matrix L_0 is in general non-singular; hence we have, in the general case,

$$D^{-1}(DF_0L_0)L_0^{-1} = F_0. \qquad (18)$$

From (18) it is evident that the rank of DF_0L_0 cannot be lower than the rank of F_0, so that, in the general case, the rank of DF_0L_0 is the same as the rank of F_0. As the matrix F_0 in Table 1 has $r + l$ linearly independent columns and n rows, the rank of F_0 is equal to the smaller of the numbers n and $r + l$. As this is also the rank of DF_0L_0, and as the inequalities $n > r$ and $r + l > r$ always hold, the number of common factors has been increased by

selection, but it cannot exceed the number of the original common factors by more than l. We then show that the additional factors appear only in the selection tests so that the number of common factors of the non-selection tests is not increased. The new, after-selection common-factor matrix of the non-selection tests is $D_t F_t L_C$, as is evident from (17) and from what was said above. As a product matrix cannot have a higher rank than its factors, the rank of $D_t F_t L_C$ cannot be higher than the rank of F_t . But this is to say that the number of common factors of the non-selection tests is not increased by the selection, and that the additional factors appear only in the selection tests. And it follows from this that in univariate selection the single additional factor, appearing in the single selection test, can be interpreted as a unique factor of this test. Thus it is possible, in univariate selection, to keep the number of common factors invariant as regards the selection test also. To sum up, we come to the Thomson-Ledermann theorem (see 5, 6, and 7):

INVARIANCE THEOREM I: *The number of common factors is not increased by selection except in the selection tests, in which there appear at most as many additional factors as there are selection tests. In univariate selection the number of common factors is not increased as regards the selection tests either.*

Our next concern will be the invariances appearing in the factor loadings under selection. As our intention is not to confine ourselves merely to the case of orthogonal factors, we first rotate the common-factor matrix F into an arbitrary common-factor matrix in the following manner. We supplement the factor matrix F, in the way shown in Table 3, by an arbitrary matrix of order $r \times r$, and normalize the rows of the last mentioned matrix. We thus obtain the matrix T. After having made sure that the rows and columns of T are linearly independent, we can regard the elements of T as the loadings of some rotated factors in terms of the original orthogonal factors. Correlations between the tests and the rotated factors are, in virtue of the basic theorem of factor analysis,

$$S = FT'. \tag{19}$$

The loadings of the rotated factor matrix S, thus obtained, do not remain invariant, but are changed by selection in the same manner as are other correlations. But we need not express the loadings of the tests in terms of the rotated factors as the correlations S. We can express the tests also as linear combinations of the factors by writing

$$F = AT \tag{20}$$

(see 1, 353) and solving for the unknown A:

$$A = FT^{-1}. \tag{21}$$

selection, but it cannot exceed the number of the original common factors by more than 1. We thus know that the additional factor comes only in the selection, [illegible] that the number of common factors of the non-selected tests is not increased. [illegible] the non-selection tests [illegible] (17) [illegible] and above [illegible] the rank of [illegible] cannot be smaller than the rank of [illegible]. And this is to say that the number of common factors of the [illegible] is not increased by the selection, and that the additional [illegible] in the selection tests. [illegible] the single additional factor [illegible] as a [illegible] factor of this test. [illegible] the number of common factors [illegible] the selection test [illegible] (see 2, [illegible], and 7).

[illegible] *Theorem* [illegible]. *The number of common factors is not* [illegible] *by selection* [illegible] *the selection tests* [illegible] *the number of common factors is not increased by* [illegible].

The next theorem will be the [illegible] selection. [illegible] the [illegible] factors, we first rotate the [illegible] matrix F into an arbitrary [illegible] factor matrix by the following [illegible]. [illegible] the factor matrix F, [illegible] of order r [illegible] the rank of the [illegible] matrix. We then obtain the matrix T. [illegible] T as the [illegible] of [illegible] factors in terms of the original [illegible] factors. [illegible] the [illegible] factors [illegible].

$$F = FT. \tag{19}$$

The [illegible] of the [illegible] factor matrix F [illegible] and [illegible] factors [illegible] by [illegible] in the [illegible] as [illegible] other [illegible]. [illegible] the [illegible] of the [illegible] in terms of the [illegible] factors [illegible] T. We [illegible] the [illegible] of the [illegible] by writing

$$F = \Lambda T \tag{20}$$

[illegible] (20) and solving for the unknown [illegible]

$$\Lambda = FT' \tag{21}$$

Finally, we partition the matrix A into submatrices A_t and A_s in accordance with the equations:

$$F_t = A_tT, \tag{22}$$

$$F_s = A_sT, \tag{23}$$

and show that the factor loadings of A_s display a given invariance in the selection of population.

As already mentioned, $D_sF_sL_C$ is the new, after-selection common-factor matrix of the non-selection tests, its elements being accordingly the after-selection loadings of the non-selection tests in terms of orthogonal factors. Analogously, it is evident (see Tables 2 and 3) that the elements of D_tTL_C are new, after-selection loadings of the—not necessarily orthogonal—rotated factors in terms of the orthogonal factors, D_t being a diagonal matrix normalizing the rows of TL_C . As a matter of notation we put

$${}_sF_s = D_sF_sL_C \, , \tag{24}$$

$${}_sT = D_tTL_C \, . \tag{25}$$

It follows from (25) that the rank of ${}_sT$ cannot be higher than the rank of L_C . On the other hand, a diagonal matrix always has an inverse, and as we had made sure that the rows and columns of T were linearly independent, T also has an inverse. Thus we can set up the equality

$$T^{-1}D_t^{-1}\,{}_sT = L_C \, , \tag{26}$$

from which it is evident that the rank of ${}_sT$ cannot be lower than the rank of L_C either. Hence, the rank of ${}_sT$ is the same as the rank of L_C . As L_C is, according to Table 2, of order $r \times (r + l)$, its rank is, in the general case, r. Consequently, the rank of ${}_sT$ is also r in the general case so that, in the general case, the rows ${}_sT$ are linearly independent. But this is to say that, in the general case, we have the equality

$${}_sF_s = {}_sA_s \, {}_sT, \tag{27}$$

and that we are able to express the non-selection tests after selection also as linear combinations of the rotated factors. ${}_sA_s$ in (27), which has so far been unknown, can be expressed in terms of A_s , i.e., the matrix corresponding to it prior to the selection, in the following way: From (23) and (24) we have first

$${}_sF_s = D_sA_sTL_C = D_sA_sD_t^{-1}D_tTL_C \, . \tag{28}$$

From this it follows, in virtue of (25),

$${}_sF_s = (D_sA_sD_t^{-1})\,{}_sT. \tag{29}$$

Comparing (27) and (29) we find

$$ {}_sA_k = D_k A_k D_t^{-1}. \tag{30}$$

As both D_k and D_t, and hence also D_t^{-1}, are diagonal matrices, it is evident from (30) that the new, after-selection factor matrix ${}_sA_k$ is obtainable from the original factor matrix A_k by multiplying the rows and columns of the latter by given numbers. A simpler kind of invariance of the factor loadings is encountered if it is assumed that the after-selection factor analysis is carried out with the covariances rather than the correlations as the starting point. According to (16) $F_k L_C$ is the new, after-selection common-factor matrix of the non-selection tests, provided that this after-selection analysis has been based on covariances. On the other hand, premultiplying (27) by D_k^{-1}, we obtain

$$D_k^{-1}{}_sF_k = D_k^{-1}{}_sA_k\,{}_sT, \tag{31}$$

which gives, in virtue of (24) and (30),

$$F_k L_C = (A_k D_t^{-1})_s T. \tag{32}$$

Consequently, $A_k D_t^{-1}$ is now the new, after-selection factor matrix corresponding to the above ${}_sA_k$. Since D_t^{-1} was a diagonal matrix, $A_k D_t^{-1}$, is obtainable from the former factor matrix, A_k, by multiplying the columns of the latter by given numbers. In other words, the columns of the after-selection factor matrix $A_k D_t^{-1}$ are proportional to the columns of the original factor matrix A_k. Thus we have proved the following:

INVARIANCE THEOREM II: *When expressing the tests after selection as linear combinations of the same common factors as before selection, the columns of the new factor matrix are proportional to the columns of the original factor matrix arrived at in the same way, provided that* (1) *the after-selection factor analysis has been carried out on the basis of covariances; and that* (2) *the selection tests have been excluded.*

We then assume that the factor matrix A_k is one representing a "simple structure." This is to say (1, 335) that (1) every row of A_k contains at least one zero, (2) every column of A_k contains at least r zeros for tests that are linearly independent, and (3) for each pair of columns there are several tests whose element in the one column is zero, but in the other column nonzero.

It was stated above that the new, after-selection factor matrix ${}_sA_k$ is obtainable from the original matrix A_k, in virtue of (30), by multiplying the rows and the columns of the latter by given numbers. Hence, the zero loadings are exactly in the same places in ${}_sA_k$ as in A_k. From this and the above definition of a simple structure there follows directly:

INVARIANCE THEOREM III: *If that part of a test battery which is left when the selection tests are excluded reveals a simple structure, this simple structure is invariant under selection in the general case.*

Selection tests have been excluded from Invariance Theorems II and III, since these tests do not reveal the kind of invariance with which these theorems are concerned. The task of proving that this is the case, we leave, however, to the reader and will confine ourselves to another circumstance connected with these two invariance theorems.

Our Invariance Theorem III is stated with the proviso that the invariance concerned appears only in the "general case." We had to make this reservation owing to the fact that this theorem was based on equation (30), which, in turn, was a consequence of (27). As was emphasized when the equality (27) was established, it is valid only in the general case when the rank of the matrix of transformation L_C , and consequently also the rank of ${}_sT$, is r. In the exceptional case when this rank is less than r, there is linear dependence among the rows of ${}_sT$. As the T-factors have been rendered linearly dependent in selection, it is not possible any more to express the tests, after selection, as linear combinations in the manner presupposed in (27).

We shall show now that this exceptional case, when the rank of ${}_sT$ and consequently also the rank of the transformation matrix L_C is less than r, may happen if the selection is "total" in some of the tests. Selection is called total in test j if the new, after-selection variance of j, s_j , vanishes (see 1, 443). On the other hand, of course, this new variance, s_j , is the jth element of the principal diagonal of the matrix $V_{jj} = V_{jj}^*$. If we put $F_j = \| c_{jm} \|$, $U_j = \| u_{jj} \|$, $L_C = \| l_{mp} \|$, and $L_U = \| l_{sp} \|$, we have from (16) the condition of total selection in the form

$$s_j = \sum_p \left(\sum_m c_{jm} l_{mp} + u_{jj} l_{jp}\right)^2 = 0. \tag{33}$$

Provided that $u_{jj} = 0$, this is reduced to

$$\sum_p \left(\sum_m c_{jm} l_{mp}\right)^2 = 0 \tag{34}$$

or

$$\sum_m c_{jm} l_{mp} = 0 \quad (p = 1, 2, \cdots, l). \tag{35}$$

The equations (35) give expression to linear dependence among the rows of $L_C = \| l_{mp} \|$, in which case the rank of L_C is reduced and, consequently, less than r. On the other hand the condition $u_{jj} = 0$ means that the communality of the test j is unity. Thus, we have proved:

INVARIANCE THEOREM IV: *If the selection is total in a test whose communality is unity, simple structure is not invariant under selection, and the tests can no longer be expressed as linear combinations of the same factors as before selection.*

Thurstone (1, 440–472) has not paid attention to the condition $u_{jj} = 0$, being of the opinion that a simple structure as a whole will never be invariant under total selection.

III. *Numerical Example*

Thomson and Ledermann (5, 6, and 7) have given numerical examples illuminating our Invariance Theorem I. Thurstone again has provided examples for our Invariance Theorems III and IV (see 1, 440–472). In order to illustrate our Invariance Theorem II as well, we make use of a numerical example constructed by Thurstone.

When factoring a given correlation matrix, the matrix F shown in Table 4 was obtained by Thurstone. We now form the matrix T by normalizing the three first rows of F. We use the pivotal condensation method to find the inverse of T, which gives us T^{-1} in Table 5. From Tables 4 and 5 we

TABLE 4*

	I	II	III
1	334	-340	688
2	544	634	048
3	543	-425	-474
4	638	379	385
5	561	023	619
6	637	-538	-067
7	561	-507	359
8	742	-051	-387
9	742	337	-196
10	820	-075	151

*Decimals omitted in all tables.

TABLE 5

430	647	665
-409	759	-527
830	061	-533

TABLE 6

	A	B	C
1	837	000	-001
2	015	837	002
3	014	000	837
4	439	724	020
5	746	418	031
6	439	001	743
7	747	000	449
8	019	418	725
9	019	724	421
10	509	483	505

TABLE 7

	I	II	III
1	265	-270	546
2	542	632	047
3	542	-424	-473
4	620	448	313
5	566	116	555
6	620	-519	-165
7	566	-514	257
8	740	-051	-386
9	740	336	-195
10	815	-024	047

TABLE 8

	A	B	C
1			
2	014	833	003
3	014	000	836
4	343	760	010
5	657	488	020
6	342	-003	774
7	661	-008	510
8	023	417	764
9	019	722	420
10	399	512	530

TABLE 9

	A	B	C
1			
2	014	833	003
3	014	000	836
4	330	730	009
5	574	427	017
6	328	-002	743
7	579	-007	448
8	023	417	764
9	019	722	420
10	378	485	501

can calculate the matrix $A = FT^{-1}$, shown in Table 6 [cf. equation (21) above]. We then select a new population so that the standard deviation of test 1 is changed from unity to .60. The intercorrelations of the tests in this new population have been computed by Thurstone according to Aitken's formulas. In addition he has also carried out a factor analysis on the basis of this new, after-selection correlation matrix and obtained the new factor matrix ${}_sF$ shown in Table 7. By normalizing the three first rows of ${}_sF$ we obtain a factor matrix ${}_sT$, and it turns out in this case that ${}_sT$ is identical with T. We therefore obtain the matrix ${}_sA_k$ in (27) as a product, ${}_sF_kT^{-1}$, which can be computed from Tables 7 and 5. The new factor matrix ${}_sA_k$ which is attained is shown in Table 8. To perform the after-selection factor analysis on the basis of covariances instead of correlations, we multiply the rows of ${}_sA_k$ by the new standard deviations of the tests. This multiplication yields the matrix $A_kD_t^{-1}$ appearing in (32). As the new standard deviations

of the tests 2, 3, $\cdots$, 10 are 1.00, 1.00, .960, .875, .960, .875, 1.00, 1.00, and .946, the resulting matrix $A_t D_t^{-1}$ is that shown in Table 9. According to our theory the columns of the matrix of Table 9 and the matrix of Table 6 should be proportional, provided that selection test 1 is excluded. As a matter of fact, this is the case, for the matrix of Table 9 is obtained from that of Table 6 by multiplying the columns of the latter by 1.315, 1.00, and 1.00, respectively.

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of the tests 1, 2, ... 10 [illegible] the resulting matrix [illegible] is then shown in Table [illegible]. [illegible] the columns of the [illegible] of Table [illegible] and the [illegible] of Table [illegible] [illegible] respectively.

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THE SQUARE ROOT METHOD AND MULTIPLE GROUP METHODS OF FACTOR ANALYSIS

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The square root method for the solution of a set of simultaneous linear equations or the reduction of a matrix has been known for some time under a variety of names. Because of its usefulness in statistical work, especially in factor analysis, the square root method is presented in general terms and an example given. Several independently developed "multiple group methods" for factor analysis are compared and synthesized. Their fundamental concepts are set forth and an appropriate system of notation developed. Detailed computational procedures are outlined, and the square root method is emphasized as a computing aid in multiple group analysis.

There are two principal objectives of this paper: (1) a clarification and coordination of several approaches to multiple group methods of factor analysis; and (2) the application of the square root method to the multiple group method of factoring a correlation matrix. In the process of meeting these objectives, a brief historical sketch of these techniques will be presented and the formal procedures will be developed and illustrated.

1. *Historical Note*

While no attempt is made to give a complete and exhaustive account of the history of the multiple group methods of factor analysis, a short account of the highlights in this development seems to be in order. Recently, Guttman (8) called attention to the basic work he had done in multiple group methods of factor analysis, which work apparently was overlooked by Thurstone (18) in his account of the similarity between the "simple method of factor analysis" proposed by Holzinger (11) and the "multiple group method of factoring the correlation matrix" which he proposed (16). A search of the literature discloses definite evidence of work in this direction prior to any of these papers.

It matters little "who got there first," but there are indications that in 1937 Horst (13) anticipated the multiple group method of factor analysis. His was a theoretical presentation, however, and the lack of computational procedures apparently was the reason that the method was not adopted and developed. Several other writers dealt with group factors in the early 1930's. Notable among these were Cyril Burt, who considered the "group factor

*The opinions expressed in this paper are those of the author and do not necessarily reflect official Department of the Army policy. The author is now with The RAND Corporation, Santa Monica, Calif.

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1. [illegible]

[illegible] the history of the multiple group methods [illegible] of factor analysis [illegible] proposed by [illegible] [11] [illegible]

[illegible]

[illegible]

[illegible]

method"; R. C. Tryon, who proposed "cluster analysis; and K. J. Holzniger, who developed the "bi-factor" method of analysis. While these methods certainly involve the group factor concept, they are not specifically in the spirit of the multiple group methods of factor analysis under consideration in the present paper. Guttman's presentation (7) of the theory in 1944 met with the same fate as Horst's paper, and for the same reasons. However, when Holzinger (11), in the same year, and Thurstone (16), a year later, presented simple computing procedures for "group factor analysis," there was ready acceptance, even though the similarity of methods was not recognized for several years (17, p. 171, 18, 8).

The history of the development and use of the "square root method" is even more vague. Certainly, as a formal mathematical procedure for the solution of a set of simultaneous linear equations or the reduction of a matrix, it must have been discovered over and over again, and may go back to the time of Gauss. Perhaps the earliest application of a square root method to the solution of normal equations in least squares theory was made by Commandant A. L. Cholesky of the French Navy around 1915, and published after his death by Commandant Benoit (3) in 1924. It was rediscovered by Banachiewicz (1, 2) in 1938 and presented as an efficient means for solving a system of linear equations and for the calculation of determinants and their inverses. The square root method was introduced in the American statistical literature in 1944 by Dwyer (4), who emphasized its use in correlation and regression (5) and who showed the relationship of this method to other methods of linear computation (6).

Concurrent with this development of the square root method as a means of solving formal mathematical and statistical problems, essentially the same technique was being devised specifically for factor analysis. It was recognized that a factor analysis of a set of variables whose intercorrelations constituted a symmetric matrix could always be obtained by a general algebraic procedure known as "completing the square." The method was applied specifically to a correlation matrix by McMahon (14) prior to 1923. Then during the rapid development of factor analysis theory in the 1930's, it was independently developed as the "diagonal method" by Thurstone (15, p. 78) and as the "solid staircase method" by Holzinger (10). Since these methods were designed expressly for factor analysis, they did not present general computing techniques. Nevertheless, they are special instances of the square root method, with the broader implication of a technique for linear computations in general.

2. *The Square Root Method*

While the square root method has been featured in several papers and applied to a variety of statistical problems during the past decade, it still is relatively unfamiliar to many researchers who use the Doolittle worksheets. As pointed out by Dwyer (6, p. 115), the advantages of the square root

method over the Gauss-Doolittle method are that it is more compact, requiring less recording, and that it permits greater ease in finding the entries to be used. Not only is the square root method more expedient for solving a symmetric set of equations, but it is especially useful in obtaining the inverse matrix in solving problems in statistics.

For the foregoing reasons, the square root method will be presented here in some detail. The procedure will be outlined in general terms and illustrated with a simple problem (9) involving the least-squares prediction of a dependent variable from three independent variables, viz.,

$$z_4 = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 \,, \tag{1}$$

where β_i is used for the conventional $\beta_{4i.(2)}$, in which the number in parentheses merely shows how many variables are held fixed. The normal equations in this case are:

$$\begin{aligned} r_{11}\beta_1 + r_{12}\beta_2 + r_{13}\beta_3 &= r_{14} \\ r_{21}\beta_1 + r_{22}\beta_2 + r_{23}\beta_3 &= r_{24} \\ r_{31}\beta_1 + r_{32}\beta_2 + r_{33}\beta_3 &= r_{34} \,, \end{aligned} \tag{2}$$

where, of course, the conditions for symmetry $r_{ij} = r_{ji}$ are satisfied and $r_{ii} = 1$ for $i, j = 1, 2, 3$.

The computing procedure for the determination of the regression coefficients in equation (1) is indicated in general terms in Table 1, and illustrated with specific numerical data. The step-by-step procedure, immediately following, is readily extended to any number of variables.

Step 1. Enter the intercorrelations among the independent variables and their correlations with the dependent variable on the first three lines of the Work Sheet.

Step 2. Obtain the sums by rows, i.e.,

$$t_j = \sum_{i=1}^{4} r_{ji} \,. \qquad (j = 1, 2, 3)$$

Note: Entries in Check column for Lines 1, 2, 3 are described in Step 10.

Step 3. The first step of the square root method is now applied by using r_{11} as a pivot. The first element in Line 4 is given by

$$s_{11} = \sqrt{r_{11}} \,,$$

while the remaining elements are obtained by the formula:

$$s_{1i} = \frac{r_{1i}}{s_{11}}. \qquad (i > 1)$$

Note: Since $r_{11} = 1$, the elements of Line 4 are equal, respectively, to the elements of Line 1.

TABLE 1

The Square Root Method

Line	Independent Variables			Dependent Variable	Total	Check
	z_1	z_2	z_3	z_4		
				General Solution		
1	r_{11}	r_{12}	r_{13}	r_{14}	t_1	r'_{14}
2	*	r_{22}	r_{23}	r_{24}	t_2	r'_{24}
3	*	*	r_{33}	r_{34}	t_3	r'_{34}
4	s_{11}	s_{12}	s_{13}	s_{14}	s_{1t}	s'_{1t}
5		$s_{22\cdot 1}$	$s_{23\cdot 1}$	$s_{24\cdot 1}$	$s_{2t\cdot 1}$	$s'_{2t\cdot 1}$
6			$s_{33\cdot 12}$	$s_{34\cdot 12}$	$s_{3t\cdot 12}$	$s'_{3t\cdot 12}$
7	β_1	β_2	β_3		$R^2_{4\cdot 123}$	$R_{4\cdot 123}$
				Numerical Illustration		
1	1.000	.693	.216	.571	2.480	.571
2	*	1.000	.295	.691	2.679	.691
3	*	*	1.000	.456	1.967	.456
4	1.000	.693	.216	.571	2.480	2.480
5		.721	.202	.410	1.332	1.333
6			.955	.262	1.217	1.217
7	.171	.492	.274		.563	.750

*Terms below the diagonal of a symmetric matrix are deleted for simplicity. Terms below the diagonal of the "square root" matrix are actually zero, and are simply omitted.

Step 4. The calculation in the Total column of Line 4 is carried out as for any other column, yielding s_{1t} . This value should agree, except for rounding errors, with the sum s'_{1t} (Check column) of all elements computed in Step 3.

Step 5. The formulas for the elements of Line 5 are:

$$s_{22\cdot1} = \sqrt{r_{22} - s_{12}^2}\,,$$

$$s_{2i\cdot1} = \frac{r_{2i} - s_{1i}s_{12}}{s_{22\cdot1}}. \qquad (i > 2)$$

Step 6. Check Line 5 by comparing the calculated value, $s_{2t\cdot1}$, with the row sum, $s'_{2t\cdot1}$.

Step 7. The formulas for the elements of Line 6 are:

$$s_{33\cdot(2)} = \sqrt{r_{33} - s_{13}^2 - s_{23\cdot1}^2}\,,$$

$$s_{3i\cdot(2)} = \frac{r_{3i} - s_{1i}s_{13} - s_{2i\cdot1}s_{23\cdot1}}{s_{33\cdot(2)}}, \qquad (i > 3)$$

where the notation $s_{3i\cdot(2)}$ is used instead of the specific $s_{3i\cdot12}$ to suggest an easy generalization when the number of variables already eliminated is more than 2.

Step 8. Apply row sum check to Line 6.

Step 9. The values of the regression coefficients are obtained by application of the following formulas (back solution):

$$\beta_3 = \frac{s_{34\cdot12}}{s_{33\cdot12}},$$

$$\beta_2 = \frac{s_{24\cdot1} - s_{23\cdot1}\beta_3}{s_{22\cdot1}},$$

$$\beta_1 = \frac{s_{14} - s_{13}\beta_3 - s_{12}\beta_2}{s_{11}}.$$

Step 10. A check on the entire computations can be made by substituting the regression coefficients back into the normal equations (2). The results are designated by r'_{14} , r'_{24} , r'_{34} and should agree (except for rounding errors) with the original correlations of independent with dependent variables.

Step 11. The multiple correlation coefficient can be computed by use of the usual formula involving the β's and r's, viz.,

$$R^2_{4\cdot123} = \beta_1 r_{14} + \beta_2 r_{24} + \beta_3 r_{34}\,.$$

From the formal solution of the three-variable problem it can be verified that

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} s_{11} & 0 & 0 \\ s_{12} & s_{22\cdot 1} & 0 \\ s_{13} & s_{23\cdot 1} & s_{33\cdot 12} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22\cdot 1} & s_{23\cdot 1} \\ 0 & 0 & s_{33\cdot 12} \end{bmatrix}. \tag{3}$$

More generally, the square root method can be formulated in matrix notation, as follows:

$$R = S'S, \tag{4}$$

whence the term "square root of a matrix" is seen to correspond to the ordinary square root of an algebraic expression. [The identity of equation (4) with the fundamental theorem of factor analysis (**12**, p. 19; **15**, p. 70) is a clear indication of why factor analysis independently discovered the square root method, although it was referred to by various names.] In other words, the square root method applied to a matrix R yields a matrix S such that premultiplication by its transpose (i.e., column-by-column multiplication of S by itself) reproduces the matrix R. It is convenient, at times, to refer to the "square root operation," by which is meant $(S')^{-1}$, since $(S')^{-1}$ operating (premultiplying) on R produces S. Then the square root operation can be applied to other matrices than the basic one from which it is derived.

3. *Concepts and Notation in Multiple Group Methods*

The square root method was presented first so that the technique could be employed as needed in the development of the multiple group methods of factor analysis. However, *the notation of the preceding section will not be used in the remainder of this paper*. Instead, a system of notation will be employed which is both clear and, when possible, closely related to that in the existing literature. The notation along with the basic concepts in multiple group factoring methods will be presented in this section.

Of course the fundamental entity in factor analysis, aside from the scores themselves, is the matrix of observed correlation coefficients. In most methods of factor analysis, the reduced correlation matrix is used, meaning that the values in the principal diagonal are reduced from self correlations of unities to (estimates of) communalities.

The basic concept that distinguishes the methods of factor analysis under consideration is that of grouping of variables. Either by arbitrary or carefully selected grouping of variables a number of common factors can be extracted in one operation, and thereby a substantial reduction in the labor of computing residual matrices is realized. Some of the different points of view regarding the selection of groups of variables will be indicated in the final section of this paper. In any event, all the group factor methods have

TABLE 2

Notation used in Multiple-Group Methods of Factoring a Correlation Matrix

Concept	In this Paper	Holzinger (11)	Thurstone (17)	Guttman (8)
Original Correlations (reduced correlation matrix)	R	R	R	G
Grouping of Variables	Groups: G_k $(k = 1, 2, \ldots, m)$	Sections: s	Groups: p	X (s common factors)
Sums of Correlations of Variables with Groups	t_{ik} $(i = 1, 2, \ldots, n)$	T_j's	S	F_o'
Sums of Correlations Among Groups	T_{jk} $(j, k = 1, 2, \ldots, m)$	T's	T	L_o
Oblique Factor Structure	$S = [s_{ik}]$	S_{js}	V	F
Correlations Among Factors	$\Phi = [r_{\gamma_j \gamma_k}]$	$r_{c_1 c_2}$	R_{pq}	L
Oblique Factor Pattern	$P = [b_{ij}]$	A_{js}	—	—
Reproduced Correlations	$R^{\dagger}$	$R^{\dagger}$	R	G_o
Residual Matrix	R_k	R_s	R_p	G_1
Transformation Matrix	Λ^{-1}	—	Λ^{-1}	—
Orthogonal Factor Matrix	$F = [a_{ik}]$	—	F	—

the common attribute of expediting the total factor analysis by the selection of a number of linearly independent groups approximating the rank of the reduced correlation matrix.

Except in rare circumstances, the common factors extracted in a single operation are oblique to one another. Therefore another basic concept is that of the matrix of correlations among the factors. Also, since the factors are correlated, the immediate results of a group factor analysis must lead to two matrices—a factor pattern and a factor structure (**12**, p. 16). The first of these gives the coefficients of the factors in the linear descriptions of the variables, while the second gives the correlations of the variables with the factors.

These results of the group factor analysis can be used to obtain a matrix of reproduced correlations; and hence the residual matrix can be determined. If the residual matrix is not sufficiently close to the null matrix then the group factor method can be applied, again, to the residual matrix.

After it has been determined that the multiple group factor solution adequately reproduces the observed correlations, some investigators may still consider such a solution as a preliminary step to the rotational problem (**17**, p. 171). In seeking "simple structure" by rotation of axes, the problem is simplified if an orthogonal frame of reference is first obtained. Hence, two additional concepts are introduced—an orthogonal factor matrix and the transformation matrix from the oblique to this orthogonal solution. The implications of stopping with the oblique solution obtained directly by the multiple group method or proceeding to the rotational problem will be brought out in the final section.

The foregoing concepts, which arise in the multiple group methods of factoring, are summarized in Table 2. The symbol associated with each concept as it will be used in the following section is listed for ready reference. Also, the notation employed by the three principal contributors to multiple group methods of factor analysis is presented to assist in making comparisons.

4. *Computations in Multiple Group Methods*

While there are apparent differences in the several presentations of the multiple group method of factor analysis—and several real differences in the generality and breadth of the theory—there is a basic technique underlying all of them. A systematic development of such a basic multiple group method is presented in this section. The procedure is illustrated with a 9-variable example, taken from Holzinger's unpublished notes, "Detailed Outline of Simple Solution," involving a sample of 696 cases, 12 tests, and 4 factors.

a. Reduced correlation matrix. The first problem is that of determining good estimates of communalities. That problem is extraneous to the scope of this paper. [Various methods for estimating communality are discussed in (**12**, pp. 156–159 and **17**, pp. 282–318).] No special justification for the

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particular choice will be given. Actually, in the 9-variable example, the estimates were obtained by a single triad for each variable, i.e., the quotient of the product of the two highest r's for a given variable by the correlation between the two variables correlating highest with the given variable. The reduced correlation matrix for the nine variables is

$$R = \begin{bmatrix} .81 & .75 & .78 & .44 & .45 & .51 & .21 & .30 & .31 \\ .75 & .69 & .72 & .52 & .53 & .58 & .23 & .32 & .30 \\ .78 & .72 & .75 & .47 & .48 & .54 & .28 & .37 & .37 \\ .44 & .52 & .47 & .91 & .82 & .82 & .33 & .33 & .31 \\ .45 & .53 & .48 & .82 & .74 & .74 & .37 & .36 & .36 \\ .51 & .58 & .54 & .82 & .74 & .74 & .35 & .38 & .38 \\ .21 & .23 & .28 & .33 & .37 & .35 & .35 & .45 & .52 \\ .30 & .32 & .37 & .33 & .36 & .38 & .45 & .58 & .67 \\ .31 & .30 & .37 & .31 & .36 & .38 & .52 & .67 & .77 \end{bmatrix},$$

where the variables are assumed to be in sequence from 1 to 9, and the estimates of the communalities appear in the principal diagonal.

b. Grouping of variables. The analysis begins with an appropriate grouping of variables. Thurstone (17) stresses the arbitrariness of grouping the variables, while Holzinger (11) and Guttman (8) emphasize the desirability of very careful selection of variables in each group according to some *a priori* hypothesis. In the example it is assumed that the nine variables can be placed in three groups such that the common factors corresponding to them will adequately explain the data. The three groups, with their constituent variables, are as follows:

$$G_1 : (1, 2, 3), \qquad G_2 : (4, 5, 6), \qquad G_3 : (7, 8, 9).$$

It may be of some interest to note that the tests in G_1 are of verbal content, G_2 arithmetic, and G_3 spatial relations.

c. Sums of correlations. The factor solution is obtained in several steps, with the computations in this step being preliminary to the actual factorial results. The sums of the correlations of each variable with the respective variables of each group are first required. These are given by the formula:

$$t_{ik} = \sum_{h \epsilon G_k} r_{ih}, \qquad (i = 1, \cdots, n; k = 1, \cdots, m) \tag{5}$$

where the sum is on the index h and the symbol "$h \epsilon G_k$" is read "h is a variable in group G_k." Formula (5) represents nm different sums. For the example, where $n = 9$ and $m = 3$, the 27 sums t_{ik} are given in Table 3.

Next, the sums of correlations among groups are obtained by means of the formula:

$$T_{jk} = \sum_{i \epsilon G_j} t_{ik}\,, \qquad (j, k = 1, \cdots, m) \tag{6}$$

where the summation is on the index i within each group, in turn. These nine sums appear in Table 4, where an immediate check is available from the symmetry property.

d. Correlations among factors. As indicated above, the common factors obtained in a single operation of a multiple group method of analysis are oblique to one another. The factors are represented by vectors through the centroid (or, more generally, a weighted average) of the respective groups of variables. While the individual variables are in standard measure, the composites are not necessarily so. If the oblique factors are designated by γ_k $(k = 1, \cdots, m)$, this means that the variance of γ_k is not unity but has the value T_{kk} as given in Table 4; and in general, Table 4 consists of the variances and covariances among the $m = 3$ factors. Then, from the theory of correlation between two composites (12, pp. 34–37), the correlations among the factors are given by:

$$r_{\gamma_j \gamma_k} = \frac{T_{jk}}{\sqrt{T_{jj}}\,\sqrt{T_{kk}}}, \tag{7}$$

and are recorded in Table 5.

e. Oblique factor structure. The correlations of the tests with the factors—the oblique factor structure—can be obtained by application of the same theory. Any test z_i is in standard measure while a factor γ_k is a composite of such variables and is not in standard form. The structure value s_{ik} is the correlation $r_{z_i \gamma_k}$ and can be computed by the formula (12, p. 36):

$$s_{ik} = \frac{t_{ik}}{\sqrt{T_{kk}}}. \tag{8}$$

The structure matrix S for the example is given in Table 6.

f. Oblique factor pattern. To complete the solution in terms of correlated factors, the linear descriptions of the variables in terms of the factors are required as well as their correlations with the factors. The coefficients in these linear equations, i.e., the pattern values, are the coordinates with respect to the oblique (factor) axes of the points representing the variables. The factor pattern can be obtained from the known factor structure S and the correlations among the factors Φ, as follows (12, p. 327):

$$P = S\Phi^{-1}. \tag{9}$$

The bulk of work implied in formula (9) is the determination of the inverse of Φ (especially for a large number of factors). Either the Doolittle method or the square root method can be used to obtain the inverse and to syste-

TABLE 3

Sums of Correlations for Variables with Groups

G_k	i	t_{i1}	t_{i2}	t_{i3}
G_1	1	2.34	1.40	.82
	2	2.16	1.63	.85
	3	2.25	1.49	1.02
G_2	4	1.43	2.55	.97
	5	1.46	2.30	1.09
	6	1.63	2.30	1.11
G_3	7	.72	1.05	1.32
	8	.99	1.07	1.70
	9	.98	1.05	1.96

TABLE 4

Sums of Correlations Among Groups: T_{jk}

$j \backslash k$	1	2	3
1	6.75	4.52	2.69
2	4.52	7.15	3.17
3	2.69	3.17	4.98
$\sqrt{T_{kk}}$	2.60	2.67	2.23

TABLE 5

Intercorrelations of Factors: Matrix Φ

	γ_1	γ_2	γ_3
γ_1	1.00	.65	.46
γ_2	*	1.00	.53
γ_3	*	*	1.00

TABLE 6

Oblique Factor Structure: Matrix S

Variable i	s_{i1}	s_{i2}	s_{i3}
1	.90	.52	.37
2	.83	.61	.38
3	.87	.56	.46
4	.55	.96	.43
5	.56	.86	.49
6	.63	.86	.50
7	.28	.39	.59
8	.38	.40	.76
9	.38	.39	.88

matically carry out the matrix multiplication to produce the pattern matrix. This explicit computation will not be done for the illustrative example, but will be accomplished along with two other concepts as outlined in the following paragraph.

g. Square root method in multiple group analysis. A very efficient scheme for the calculation of the oblique factor pattern, the transformation matrix, and the orthogonal factor matrix can be developed by application of the square root method. The basic square root operation will be performed on the matrix Φ of factor correlations, and the resulting "square root" matrix will be denoted by Λ. Expressed symbolically,

$$\Phi = \Lambda'\Lambda, \tag{10}$$

and the square root operator is $(\Lambda')^{-1}$. This operator when applied to the transpose of the structure matrix yields the desired orthogonal factor matrix, i.e.,

$$(\Lambda')^{-1}S' = F'. \tag{11}$$

Stated another way, the orthogonal factor matrix is obtained from the oblique factor structure by means of the transformation matrix Λ^{-1}, as follows:

$$F = S\Lambda^{-1}. \tag{12}$$

The mathematical proof of the foregoing expression as the transformation from an oblique to a rectangular coordinate system will not be presented here. A brief description of this special type of transformation is in order, however. The given set of variables may be considered as points in the common factor space, for which sets of coordinates are $F = [a_{ik}]$ and $P = [b_{ij}]$ with respect to an orthogonal and an oblique frame of reference, respectively. Now, if the oblique coordinates are known, the special transformation desired is one in which the first axis of the new system coincides with the first oblique factor axis, the second is in the plane of the first two oblique axes and orthogonal to the first, etc. This transformation is accomplished directly by the square root method applied to the matrix of cosines of angular separations of oblique axes (namely, the correlation matrix Φ). The resulting transformation from oblique to the specified orthogonal coordinates is given by the matrix equation:

$$F = P\Lambda'. \tag{13}$$

But, upon substituting the expression for P from equation (9) into equation (13), it follows that

$$F = S\Phi^{-1}\Lambda' = S\Lambda^{-1}(\Lambda')^{-1}\Lambda' = S\Lambda^{-1},$$

which is precisely formula (12). Hence the transformation to the orthogonal

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frame of reference (F) is described in terms of projections (s_{ik}) on the oblique axes instead of the coordinates (b_{ik}) in the original oblique reference system.

As indicated above, the orthogonal factor matrix F as expressed in equation (11) can be obtained by application of the square root method. Also, by means of the square root method, the oblique pattern matrix P can be obtained without explicit computation of the inverse of Φ, which is required in formula (9). It follows from formula (13) that

$$P' = \Lambda^{-1}F', \tag{14}$$

and since each element in the right-hand member of this expression comes from the application of the square root method, the oblique pattern results from a simple matrix multiplication.

The schematic formulation for the square root method to get the transformation matrix, the orthogonal factor matrix, and the oblique factor pattern is presented in Table 7A, and the application to the 9-variable example appears in Table 7B. In applying this procedure to a larger set of variables, the worksheet might more conveniently be arranged to list the matrices S, F, and P in adjacent vertical blocks instead of their transposes in horizontal sections.

h. Reproduced correlations. If the multiple group method of analysis is carried to the stage of an orthogonal factor solution, then the reproduced correlations can be obtained by the fundamental theorem of factor analysis, viz.,

$$R^{\dagger} = FF'. \tag{15}$$

On the other hand, if the multiple group method of analysis is employed to get an oblique solution without rotation of axes, then equation (15) is not applicable. There are several formulas for getting the reproduced correlations directly from the component parts of an oblique solution. The basic formula (**12**, p. 19), involving the multiplication of three matrices, follows:

$$R^{\dagger} = P\Phi P'. \tag{16}$$

Two other expressions for the reproduced correlations (**12**, p. 327) are:

$$R^{\dagger} = PS' = SP', \tag{17}$$

which involve precisely the same kind of computation as formula (15). Still another formula is

$$R^{\dagger} = S\Phi^{-1}S', \tag{18}$$

which is essentially the computation described by Guttman (**8**, p. 215).

Since both the orthogonal and oblique solutions were obtained for the illustrative example, the reproduced correlations were computed by formula

TABLE 7

Square Root Method in Multiple Group Analysis

A. Schematic Form

Factors		Variables	Instructions
Factor Correlations Φ $(\Phi = \Lambda'\Lambda)$	Identity Matrix I	Transpose of Oblique Factor Structure S'	Original (Known) Martices
Square Root Matrix Λ	Transpose of Transformation Matrix $(\Lambda')^{-1}$	Transpose of Orthogonal Factor Matrix $(\Lambda')^{-1} S' = F'$	Square Root Operation $(\Lambda')^{-1}$
	Inverse $\Lambda^{-1} (\Lambda')^{-1} = \Phi^{-1}$ (not required)	Transpose of Oblique Factor Pattern $\Lambda^{-1} F' = P'$ (also $\Lambda^{-1} (\Lambda')^{-1} S' = \Phi^{-1} S' = P'$)	Col. - by - Col. Mult. with $(\Lambda')^{-1}$ (Premult. by Λ^{-1})

B. Illustrative Example

	Y_1	Y_2	Y_3				1	2	3	4	5	6	7	8	9	Total	9-Var. Total	Check
1	1.0000	.6511	.4640	1.0000	0	0	.90	.83	.87	.55	.56	.63	.28	.38	.38	8.48		
2	*	1.0000	.5324	0	1.0000	0	.52	.61	.56	.96	.86	.86	.39	.40	.39	8.74		
3	*	*	1.0000	0	0	1.0000	.37	.38	.46	.44	.49	.50	.59	.76	.88	7.86		
	1.0000	.6511	.4640	1.0000	0	0	.90	.83	.87	.55	.56	.63	.28	.38	.38	8.48	5.37	8.46
		.7590	.3034	-.8578	1.3175	0	-.08	.09	-.01	.79	.65	.60	.28	.20	.19	4.24	2.72	4.24
			.8323	-.2448	-.4803	1.2015	-.03	-.04	.07	-.07	.04	.03	.45	.63	.77	3.17	1.86	3.17
				1.7957	-1.0126	-.2941	.98	.76	.85	-.11	-.01	.11	-.08	.05	.02		2.58	2.58
				*	1.9665	-.5771	-.09	.14	-.04	1.07	.84	.77	.15	-.04	-.12		2.69	2.69
				*	*	1.4436	-.04	-.05	.08	-.08	.04	.04	.55	.76	.93		2.23	2.23

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(15) and checked by the second of formulas (17). The reproduced correlation matrix, including reproduced communalities, follows:

$$R^{\dagger} = \begin{bmatrix} .82 & .74 & .78 & .43 & .45 & .51 & .21 & .31 & .30 \\ & .70 & .72 & .53 & .52 & .57 & .24 & .31 & .30 \\ & & .75 & .47 & .48 & .54 & .27 & .37 & .38 \\ & & & .93 & .82 & .81 & .34 & .32 & .31 \\ & & & & .74 & .74 & .36 & .37 & .37 \\ & & & & & .75 & .36 & .38 & .38 \\ & & & & & & .36 & .45 & .51 \\ & & & & & & & .58 & .67 \\ & & & & & & & & .78 \end{bmatrix},$$

where the terms below the diagonal of the symmetric matrix were deleted for simplicity.

i. Residual matrix. The residual matrix, with k factors removed, is defined by

$$R_k = R - R^{\dagger}. \tag{19}$$

For the 9-variable example, the residual matrix is

$$R_3 = \begin{bmatrix} -.01 & .01 & .00 & .01 & .00 & .00 & .00 & -.01 & .01 \\ & -.01 & .00 & -.01 & .01 & .01 & -.01 & .01 & .00 \\ & & .00 & .00 & .00 & .00 & .01 & .00 & -.01 \\ & & & -.02 & .00 & .01 & -.01 & .01 & .00 \\ & & & & .00 & .00 & .01 & -.01 & -.01 \\ & & & & & -.01 & -.01 & .00 & .00 \\ & & & & & & -.01 & .00 & .01 \\ & & & & & & & .00 & .00 \\ & & & & & & & & -.01 \end{bmatrix}$$

5. *Comparison of Multiple Group Methods*

One of the principal differences that appears in the several developments of multiple group methods of factor analysis really is concerned with the formulation of scientific hypotheses rather than with the *method* of analysis. Guttman (8) and Holzinger (11) suggest that the multiple group methods be used in conjunction with some *a priori* psychological theory. Guttman

emphasizes the fact that the computational procedures of multiple group methods can be applied in any event, but most psychological meaning can be gained only through the testing by the data of preconceived hypotheses. These hypotheses are reflected in the specific manner of grouping the variables and in the resulting common factors (usually oblique).

On the other hand, Thurstone (17, pp. 171, 173) emphasizes that the multiple group method of factoring is quite independent of the manner of grouping the variables, and in an example, deliberately sets up groups in an arbitrary fashion, with little reference to the correlations. This thought is evident when Thurstone (18) calls attention to the unnecessary restrictions that Holzinger (11) placed upon the matrix of correlations in order to use his "simple method of factor analysis." These restrictions *are* unnecessary when the object is simply to get a reduction of the correlation matrix to a factor matrix by the expedient multiple group method; they are *not* unnecessary when the object is to test some specified hypothesis by use of multiple group analysis.

Thus, while Holzinger considers the multiple group method suitable only if the correlation matrix is amenable to sectioning into portions of approximate unit rank, and Guttman prefers to group the variables so as to avoid or reduce the problem of rotation of axes, Thurstone conceives of the multiple group method primarily as another (efficient) technique for initial factoring to provide an orthogonal factor matrix, "which is the starting-point for the rotational problem" (17, p. 171).

There is no doubt about the effectiveness of the multiple group methods in reducing the labor of computing residual matrices. Instead of extracting one factor at a time, and computing a residual matrix after each, the basic theorem (7, p. 12) underlying multiple group methods implies that only one residual matrix need be computed after extracting several factors at one time. If a number of linearly independent groups is selected equal to the dimension of the common factor space then only one residual matrix will be necessary; otherwise, if the first estimate of the number of clusters is too small, the process has to be repeated again. If too many clusters should be selected then the case of multi-collinearity will be evident in the matrix of correlations among the group factors (and the inverse will not exist).

It has been pointed out that if the total number of common factors is not extracted in a single operation, then the multiple group method can be applied again to the residual matrix obtained after the first operation. And this can be repeated as many times as necessary to bring the residuals down to negligible values. In the successive application of the multiple group method, the common factors extracted at each stage are oblique to one another, but the factors obtained in any single stage are orthogonal to all factors extracted in other stages. The implication of this is that if an *a priori* hypothesis involves an oblique structure then all the common factors must

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be extracted in one operation, or else subsequent rotation might be necessitated.

Both Holzinger and Thurstone have considered only simple (unit-weighted) composites of variables in essentially non-overlapping groups. Guttman (8, p. 216) considers this case as the simplest, and usually quite adequate, manner of grouping variables. However, by use of the weight matrix X, he presents the most general approach to multiple group methods of factor analysis.

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A SOLUTION FOR CASE III OF THE LAW OF COMPARATIVE JUDGMENT

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A solution for Case III of the Law of Comparative Judgment, modeled after Thurstone's solution for Case IV but eliminating the restrictive assumption of relatively equal discriminal dispersions, is developed.

Case III of the Law of Comparative Judgment (6, p. 280) may be written as follows:

$$S_k - S_j = X_{kj}\sqrt{\sigma_k^2 + \sigma_j^2}\,, \tag{1}$$

where

S_k = scale value or average momentary estimate of the position of stimulus k on the psychological continuum,
S_j = scale value or average momentary estimate of the position of stimulus j on the psychological continuum,
σ_k = discriminal dispersion or standard deviation of the distribution of momentary estimates of the position of stimulus k on the psychological continuum,
σ_j = discriminal dispersion or standard deviation of the distribution of momentary estimates of the position of stimulus j on the psychological continuum, and
X_{kj} = normal deviate corresponding to the proportion of times that stimulus k is chosen instead of stimulus j when the two are compared.

The assumptions leading to Case III (6, pp. 273–280) may be stated as follows:

(a) Whenever a subject is about to choose one of two stimulus objects on some basis controlled by the experimenter, each object arouses its own momentary estimate of the true scale value of that stimulus on the uni-dimensional psychological continuum which is the basis of the choice.

(b) Of two stimuli being compared, the one which has the largest momentary estimate is the one chosen by the subject.

(c) The distribution of differences between the momentary estimates for any two stimuli is normal in form. (A careful following of Thurstone's development shows that this is the only normality assumption that is required, although it has been customary to assume that the distribution of momentary estimates for each stimulus is normal in form.)

(d) The correlation between the momentary estimates for any two stimuli is zero.

Equation (1) could be written explicitly for all values of j and k to form a set of simultaneous equations in the S and σ values. Probably because of their non-linear form, however, no solution for this set of simultaneous equations has appeared. A section in Guilford entitled "Solution by Case III" (**4**, pp. 233–234) may seem to imply that such a solution already exists, but the text of this section indicates otherwise. The heading merely refers to the determination of the scale values from equation (1) after the discriminal dispersions have been estimated by other means.

By making use of the additional assumption that

(e) The discriminal dispersions are all of the same order of magnitude, Thurstone (**6**, pp. 280–281) has developed what is called Case IV of the Law of Comparative Judgment, as follows:

$$S_k - S_j = \frac{X_{kj}}{\sqrt{2}} (\sigma_k + \sigma_j). \tag{2}$$

Since equation (2) is a homogeneous linear equation in the S and σ values, any method of solving such equations could be applied to reach a solution for Case IV. A least squares solution has recently been provided by Gibson (**3**). Thurstone (**7**, pp. 293–296) earlier worked out an ingenious and rapid solution based on a consideration of the analytic geometry involved in the plots of columns of the square matrix of X values. His estimate of σ_k is

$$\sigma_{ak} = (a/V_k) - 1, \tag{3}$$

where

$$a = \frac{2N}{\sum\limits_k (1/V_k)}, \tag{4}$$

V_k being the standard deviation of the entries in column k of the matrix X (containing the X values). His unit of measurement is defined as follows:

$$\sum_k \sigma_k \equiv N. \tag{5}$$

The solution for Case III to be developed here is modeled after Thurstone's solution for Case IV but does not involve the restrictive assumption of relatively equal discriminal dispersions. A recent article by Burros (2) presents a solution for a combination of Cases III and IV which nevertheless retains the assumption of approximately equal discriminal dispersions. This is shown by Burros' use of his equation (17) in the development of his solution. It, too, is modeled after Thurstone's solution for Case IV. Burros' estimate of σ_k is

$$\sigma_{ck} = \frac{c}{V_k^2}, \tag{6}$$

where

$$c = \frac{N}{\sum_k (1/V_k^2)}, \tag{7}$$

and V_k is defined as before. His unit of measurement is given by equation (5).

For the development of a solution for Case III, consider any three stimuli, h, j, and k. Let h and j be held constant, once chosen, and let k vary. Then equation (1) holds for stimuli j and k. A similar equation may be written for stimuli h and k:

$$S_k - S_h = X_{kh}\sqrt{\sigma_k^2 + \sigma_h^2}\,. \tag{1a}$$

Substracting equation (1a) from equation (1) and rearranging terms, we obtain,

$$X_{kj} = \frac{\sqrt{\sigma_k^2 + \sigma_h^2}}{\sqrt{\sigma_k^2 + \sigma_j^2}}X_{kh} + \frac{S_h - S_j}{\sqrt{\sigma_k^2 + \sigma_j^2}}. \tag{8}$$

Equation (8) is linear in X_{kj} and X_{kh} for a fixed value of k. Hence the plot of column j against column h from the matrix X will be approximately linear. Deviations from strict linearity are attributable to the imperfect applicability of Case III to the empirical data, sampling error, and variations in σ_k as k takes on different values.

Now some kind of average of the expression

$$m_k \equiv \frac{\sqrt{\sigma_k^2 + \sigma_h^2}}{\sqrt{\sigma_k^2 + \sigma_j^2}}, \tag{9}$$

over all values of k, should represent the slope of the best-fitting line for this set of points. Since the choice of a particular measure of central tendency is essentially arbitrary, a convenient one for the present purpose, though a little unusual, is the square root of the ratio of the arithmetic mean of the

squared numerators to the arithmetic mean of the squared denominators. Since h and j are held constant while k varies, the result is then

$$m = \frac{\sqrt{\frac{1}{N} \sum_k \sigma_k^2 + \sigma_h^2}}{\sqrt{\frac{1}{N} \sum_k \sigma_k^2 + \sigma_j^2}}. \tag{10}$$

The choice of this peculiar measure of central tendency is really more than a matter of convenience; it is essential to the present solution for Case III. This alone justifies its use here. Other considerations lose much of their importance when it is recognized that this average usually will not deviate markedly from the more conventional ones. An added guarantee of this is the fact that σ_k occurs in both numerator and denominator of equation (9), thus inhibiting wide fluctuations in the size of the fraction from one value of k to the next.

Equation (10) can be further simplified by defining the unit of measurement for the scale in the following way:

$$\sum_k \sigma_k^2 \equiv N. \tag{11}$$

Then the slope of the best-fitting line is

$$m = \frac{\sqrt{1 + \sigma_h^2}}{\sqrt{1 + \sigma_j^2}}. \tag{12}$$

The plot of column j against column h of the matrix X could actually be constructed and the slope of the best-fitting line determined by any fitting method. Thurstone (7, pp. 293–294) construed this plot as representing a scatter diagram, with the slope of the line of best fit being simply the geometric mean of the slopes of the two regression lines. This leads to the following formula for the slope of the best-fitting line:

$$m = V_j/V_h , \tag{13}$$

where V_j represents the standard deviation of the entries in column j of the matrix X and V_h represents the standard deviation of the entries in column h of the same matrix. Since equation (13) is essentially the ratio of two indices of dispersion, the V's might be less restrictively defined as being any reasonably stable measure of variation, such as the average deviation.

Equations (12) and (13) give

$$m = \frac{\sqrt{1 + \sigma_h^2}}{\sqrt{1 + \sigma_j^2}} = \frac{V_j}{V_h}. \tag{14}$$

Hence,

$$V_h \sqrt{1 + \sigma_h^2} = V_j \sqrt{1 + \sigma_j^2} , \tag{15}$$

so that

$$V_h^2(1 + \sigma_h^2) = V_j^2(1 + \sigma_j^2). \tag{16}$$

Since equation (16) holds for any values of h and j, either side may be set equal to a constant defined as b. Then, for stimulus k,

$$V_k^2(1 + \sigma_k^2) = b. \tag{17}$$

Equation (17) was previously employed by Burros (**2**, p. 64).

Now if equation (17) is written in the form

$$(b/V_k^2) = 1 + \sigma_k^2 \tag{18}$$

and then summed over k, the result is

$$b \sum_k (1/V_k^2) = N + \sum_k \sigma_k^2 . \tag{19}$$

By virtue of equation (11), equation (19) can be further simplified and then solved explicitly for b to give

$$b = \frac{2N}{\sum_k (1/V_k^2)}. \tag{20}$$

Note that b is twice the harmonic mean of V_k^2. Equation (18) can be written in the form

$$\sigma_{bk}^2 = (b/V_k^2) - 1 \tag{21}$$

to give the Case III estimate of σ_k^2 once b has been determined.

The method of solution for the discriminal dispersions is then quite direct. Compute the squared standard deviation (or the square of some other stable measure of dispersion) for the entries in each column of the matrix X, and then determine its reciprocal. Sum these reciprocals and substitute the result into equation (20) to determine b. Substitute b and the appropriate V_k^2 into equation (21) to get σ_{bk}^2 .

These σ_k^2 estimates must now be used to determine the scale values. An efficient method for doing this will be given here. First form the matrix Q, with the general entry

$$q_{kj} \equiv \sigma_k^2 + \sigma_j^2 , \tag{22}$$

by bordering the space to be occupied by Q with an extra column and row containing the σ_k^2 estimates. Then each entry in Q is simply the sum of the σ_k^2 estimates that head the row and column which intersect in that entry. The kth diagonal element in Q is $2\sigma_k^2$. A convenient summational check on the correct formation of column j in Q is to be had from the equation

$$\sum_k q_{kj} = \sum_k \sigma_k^2 + N\sigma_j^2 = N + N\sigma_j^2 = N(1 + \sigma_j^2), \tag{23}$$

which simplifies by virtue of equation (11). The next step is to form a matrix R with the general entry

$$r_{kj} \equiv \sqrt{q_{kj}} = \sqrt{\sigma_k^2 + \sigma_j^2} . \tag{24}$$

There is no summational check here, but of course each entry in R, when squared, should yield the corresponding entry in Q. The next step is to form the matrix S, whose general entry is given by equation (1) and is obtained from the matrices X and R by multiplying the kj entry in X by the corresponding entry in R. There is no summary check here. It should be pointed out explicitly that the symmetry (or skew-symmetry) of the matrices X, Q, R, and S permits a substantial reduction (by almost half) of the work that would otherwise be required to form these matrices.

Now the sum of column j in the matrix S is

$$\sum_k (S_k - S_j) = \sum_k S_k - NS_j . \tag{25}$$

Since the origin of the scale has not yet been defined, it is convenient to fix it as follows:

$$\sum_k S_k \equiv 0. \tag{26}$$

Equation (25) then reduces to

$$\sum_k (S_k - S_j) = -NS_j . \tag{27}$$

Hence,

$$S_j = \frac{-1}{N} \sum_k (S_k - S_j). \tag{28}$$

Thus the scale values themselves are obtained simply by multiplying through the column sums of the matrix S by the quantity $(-1/N)$. A check here is that, by equation (26), the sum of the resulting scale values should be zero.

The origin and unit of measurement for the resulting scale are defined by equations (26) and (11), respectively. Any desired change of unit or origin can be effected by appropriate modifications of the scale values and discriminal dispersions. The unit of measurement can be altered by applying a constant multiplier to both the scale values and the discriminal dispersions. A change of origin (to make all scale values positive, for example) can be brought about by adding a suitable constant to all scale values without changing the discriminal dispersions.

This solution for Case III requires that the matrix X be completely filled, with no blanks occurring in the side entries, and with zeros (presumably) in the diagonal cells. If some entries in an X table are missing because of the instability of normal deviates for very high or very low proportions, the table can usually be rearranged and then subdivided into a number of

overlapping square blocks in such a way that each block is complete and skew-symmetric. Then this solution can be applied to each block separately and the results for all blocks combined by some fitting procedure. These remarks apply also to the Case IV solution of Thurstone and the Cases III-IV solution of Burros.

A pre-publication reviewer has called attention to a possible empirical difficulty by exhibiting a fictitious example in which the present Case III solution yields some negative σ_{bk}^2 values. Equations (21) and (20) indicate that a negative σ_{bk}^2 necessarily arises whenever any V_k^2 exceeds the value of b, or whenever the reciprocal of any V_k^2 is less than half the arithmetic mean of those reciprocals. Equations (3) and (4) indicate a completely analogous state of affairs with Thurstone's solution for Case IV. His solution will necessarily yield a negative σ_k estimate whenever any $1/V_k$ is less than half the mean of such reciprocals. Since the rank ordering of the various values of $1/V_k^2$ is the same as that for $1/V_k$, absurd results are likely to occur for the same stimuli by both of these solutions. They may be more serious with the Case III solution because the stronger negative skewing of the distribution of $1/V_k^2$ may make it easier for the tail of that distribution to extend below half its mean.

A fictitious example which yields absurd results by both of these solutions is shown in Table 1. Table 2 shows the corresponding X matrix, as given by equation (1) solved explicitly for X_{kj}. Table 3 gives the results of applying

TABLE 1

A Fictitious Paired Comparisons Example

	Stimulus				
	1	2	3	4	5
S_k	-.24	-.24	-.04	.16	.36
σ_k	.001	2.00	1.50	1.00	.50

TABLE 2

The Fictitious X Matrix

		Stimulus				
		1	2	3	4	5
Stimulus	1	.000	.000	.133	.400	1.200
	2	.000	.000	.080	.179	.291
	3	-.133	-.080	.000	.111	.253
	4	-.400	-.179	-.111	.000	.179
	5	-1.200	-.291	-.253	-.179	.000

TABLE 3

Three Thurstone-Type Solutions

	Stimulus				
	1	2	3	4	5
V_k^2	.2034	.0125	.0192	.0368	.1762
σ_{ak}	-.146	2.444	1.778	1.007	-.083
σ_{bk}^2	-.711	3.711	2.065	.601	-.666
σ_{ck}	.144	2.356	1.532	.800	.167

$a = .3850$ $b = .05888$ $c = .02944$

to Table 2 the Case IV solution of Thurstone (σ_{ak}), the present Case III solution (σ_{bk}^2) and the Cases III-IV solution by Burros (σ_{ck}). The first two solutions give absurd results for the same two stimuli, the relative magnitude of the negative values being larger in the Case III solution. Both of these solutions apparently suffer from the same basic defect, namely, that small

positive true values may be estimated as negative by the fitting procedure that is used.

The last row in Table 3 contains a simple answer to this dilemma, if it should ever arise with empirical data. None of the Burros estimates of σ_k is negative, and equations (6) and (7) show that they never can be negative. It is therefore recommended that, in the event of negative σ_k^2 estimates by the present Case III solution, the Burros solution be used. Equations (7) and (20) give

$$c = \frac{b}{2}, \tag{29}$$

and equations (6), (29), and (20) yield

$$\sigma_{ck} = \tfrac{1}{2}(b/V_k^2) = \tfrac{1}{2}[(b/V_k^2 - 1) + 1] = \tfrac{1}{2}(\sigma_{bk}^2 + 1). \tag{30}$$

Hence all that is involved in switching from the present solution to that of Burros is to add 1 to each σ_{bk}^2 and divide by 2 to get the corresponding σ_{ck}, all of which will be positive real numbers. The only changes in the subsequent computations for the scale values, as they have been outlined here, is that the last two steps in equation (23), a check formula, are invalid because equation (11) is then replaced by equation (5) as the definition of the unit of measurement. The first equality in equation (23) still holds, however, so that a summational check still exists at that stage.

This Case III solution has been applied to several different sets of empirical data without yielding absurd results. These researches include a traditional psychophysical experiment (**5**), a study of nationality preference (**4**, p. 227), and two studies of reaction potential (**1** and **2**). It is the writers' belief that absurd results will occur only rarely with empirical data, so that this solution should have fairly wide applicability in psychological research.

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THE PREDICTED AND OBSERVED EFFECT OF CHANCE SUCCESS ON MULTIPLE-CHOICE TEST VALIDITY*

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Assuming chance to be fully operative, the predicted effect of chance success on test validity when answer options are supplied depends on the number of options, the difficulty of the test and the variance of test scores. Predicted validity values are compared with empirical validity values in an experiment which used the same mathematics test items with and without answer options.

The present paper is an extension of an earlier study on the effect of chance success on item and test statistics when answer options are supplied. The earlier study (3) compared values predicted by Guilford's formula for item difficulty (2), Carroll's formula for test reliability (1), and the author's formula for item-test correlation with values obtained in an empirical study using mathematics achievement test material. The present paper presents the equation for predicting the effect of chance success on test validity and compares predicted values with empirical values from the same source of data as that used for the previous study.

In deriving the prediction equation, the assumptions made in the earlier study will apply here. They are:

1. Every examinee attempts every item in the test.
2. Every examinee who knows the correct answer to an item answers the item correctly in both multiple-choice and answer-only form.
3. Every examinee who does not know the correct answer to an item answers the item incorrectly in answer-only form and chooses from among the options on a basis of chance alone in multiple-choice form.
4. The number of options per item is the same for all items in the multiple-choice form of the test.

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The following notations will be used:

Multiple-Choice Form of Test	Answer-Only Form of Test	
R		Probability of answering an item correctly on the basis of chance alone, where $R = 1/\text{number of answer options}$.
W		Probability of answering an item incorrectly on the basis of chance alone, where $W = 1 - R$.
h	h	Number of items in the test.
x'	x	Score (number of items correct) on the test.
x'_k	x_k	Score of the kth individual on the test.
y_k	y_k	Score of the kth individual on the criterion.
t	t	Number of individuals who take the test.
$M_{x'}$	M_x	Mean score on the test.
M_y	M_y	Mean score on the criterion.
$\sigma_{x'}$	σ_x	Standard deviation of scores on the test.
σ_y	σ_y	Standard deviation of scores on the criterion
$r_{x'y}$	r_{xy}	Correlation between scores on the test and scores on the criterion.

Carroll (1) has given formulas for the expected multiple-choice mean and standard deviation, which can be expressed in terms of number-right scores as follows:

$$M_{x'} = WM_x + Rh, \tag{1}$$

and

$$\sigma_{x'} = \sqrt{W^2\sigma_x^2 + RW(h - M_x)}. \tag{2}$$

It will be seen from (1) that

$$\sum_{k=1}^{t} x'_k = W \sum_{k=1}^{t} x_k + Rht. \tag{3}$$

If multiple-choice scores on the test for those individuals making a given score on the criterion behave according to large sampling expectations, it may be demonstrated that

$$\sum_{k=1}^{t} x'_k y_k = W \sum_{k=1}^{t} x_k y_k + Rh \sum_{k=1}^{t} y_k \,. \tag{4}$$

Substituting (1), (2), and (4) into the formula for the correlation between x' and y,

$$r_{x'y} = \frac{\frac{1}{t} \sum_{k=1}^{t} x'_k y_k - M_{x'} M_y}{\sigma_{x'} \sigma_y} \,. \tag{5}$$

it will be seen that

$$r_{x'y} = \frac{\frac{W}{t}\sum_{k=1}^{t} x_k y_k - WM_xM_y}{\sigma_y\sqrt{W^2\sigma_x^2 + RW(h - M_x)}} . \tag{6}$$

It will be noted that the numerator of (6) is equal to $Wr_{xy}\sigma_x\sigma_y$, and hence

$$r_{x'y} = \frac{Wr_{xy}\sigma_x}{\sqrt{W^2\sigma_x^2 + RW(h - M_x)}} , \tag{7}$$

or

$$\frac{r_{xy}}{r_{x'y}} = \sqrt{1 + \frac{R}{W}\left(\frac{h - M_x}{\sigma_x^2}\right)}. \tag{8}$$

It will be seen from equation (8) that the validity of the multiple-choice form of a test is theoretically always less than the validity of the same test in answer-only form. As σ_x^2 becomes very large relative to $h - M_x$ or as W approaches 1,

$$\lim \frac{r_{xy}}{r_{x'y}} = 1.$$

Thus, if the mean score for the answer-only form of a standardized test is equal to about one-half the number of items and the standard deviation of scores is equal to about one-fifth the number of items, then the multiple-choice validity will theoretically approach the answer-only validity as a limit as the number of items and/or the number of answer options becomes large.

As described in the earlier study, each of four 36-item sections of mathematics achievement material was prepared both in multiple-choice form (with five answer options) and in answer-only form. An additional 16-item section was prepared which contains both answer-only and multiple-choice items. These five sections were combined into four tests in the manner indicated in Table 1. The four 36-item sections in multiple-choice form are designated as

TABLE 1

Arrangement of Tests

	Time (minutes)	Number of Items	Test W (N = 125)	Test X (N = 130)	Test Y (N = 128)	Test Z (N = 126)
Part I	15	16	Set of items, common for all tests, with odd-numbered items in answer-only form and even-numbered items in multiple-choice form.			
Part II	35	36	Section M1	Section A3	Section M3	Section A1
Part III	35	36	Section M2	Section A4	Section M4	Section A2
Part IV	35	36	Section A3	Section M1	Section A1	Section M3
Part V	35	36	Section A4	Section M2	Section A2	Section M4

TABLE 2
Total Test Statistics

Statistic	Test W ($N = 125$) Part I	Part II or IV*	Part III or V*	Test X ($N = 130$) Part I	Part II or IV**	Part III or V**	Test Y ($N = 128$) Part I	Part II or IV*	Part III or V*	Test Z ($N = 126$) Part I	Part II or IV**	Part III or V**
Number of items	16	36	36	16	36	36	16	36	36	16	36	36
Mean score												
Mixed	10.26			9.60			10.41			9.93		
Observed answer-only		19.82	19.11		15.72	16.55		17.79	19.12		15.63	16.05
Observed multiple-choice		19.31	19.64		19.48	20.97		19.88	20.51		21.48	21.56
Standard deviation of scores												
Mixed	3.16			3.25			2.83			3.25		
Observed answer-only		5.46	5.60		5.22	5.70		5.35	5.89		4.83	6.07
Observed multiple-choice		4.49	5.62		5.08	5.87		4.79	5.39		5.22	5.34
*Reliability****												
Observed answer-only		.82			.79			.85			.82	
Observed multiple-choice		.78			.77			.71			.81	
Validity—Mathematics												
Mixed	.45			.52			.53			.61		
Observed answer-only		.54	.57		.60	.57		.62	.57		.64	.64
Observed multiple-choice		.54	.54		.62	.66		.55	.49		.65	.62
Predicted multiple-choice		.51	.54		.55	.53		.58	.53		.58	.60
Validity—Academic standing												
Mixed	.39			.50			.50			.53		
Observed answer-only		.49	.50		.60	.55		.58	.54		.57	.58
Observed multiple-choice		.50	.47		.62	.64		.55	.44		.60	.56
Predicted multiple-choice		.46	.47		.55	.51		.54	.51		.51	.55

*II, III are multiple-choice; IV, V are answer-only. **II, III are answer-only; IV, V are multiple-choice.

***Based on correlation between scores on separately timed parts.

M1, M2, M3, and M4; the same sections in answer-only form, as A1, A2, A3, and A4, respectively. Thus, Tests W and X, and again Tests Y and Z, differed only in the order in which the multiple-choice and answer-only sections were administered.

In order to compare the extent to which the predicted effects of chance success on test validity agree with results obtained in practice, rank standing in first-year mathematics and total academic standing in all courses for the first year were obtained for the subjects used in the study. Validities were computed against both criteria for the 509 subjects for whom complete data were available. Results are shown in Table 2.

It will be noted that equation 8 tended to over-correct and that values obtained by applying equation 8 to the observed answer-only values are no nearer than the observed answer-only values themselves to the observed multiple-choice values. Moreover, the observed answer-only validity values are quite similar to observed multiple-choice validity values; no statistical difference was found among the 16 validity coefficients for either criterion. Although the differences between predicted and obtained multiple-choice validity values are not great, the direction of the differences appears to be significant by a simple signs test. From the data in Table 2, then, it cannot be concluded that the multiple-choice form of test predicts less well than the answer-only form.

One may conclude, therefore, on the basis of the current investigation, that:

1. Theoretically, supplying options for a test decreases its validity.
2. For the mathematics achievement tests used in the current study, the multiple-choice form was as effective as the answer-only form for predicting success in coursework, even though the tests were fairly short.

It will be noted that the findings in the current study regarding validity thus correspond to those regarding reliability in the earlier study. As in the case of multiple-choice test reliability, some explanation for the deviation of observed multiple-choice validity from predicted multiple-choice validity may be found by considering the extent to which the assumptions basic to the prediction formulas were met in practice.

The fourth assumption was met; all items had five options. Although the first assumption was not met, a comparison of item statistics for items not reached by examinees with those reached by most examinees revealed no differentiating pattern which might influence the validity results.

The third assumption was likely not met since examinees made careless errors; the greater likelihood that such errors will be detected in multiple-choice form and rectified may contribute to increased validity for the multiple-choice answer form.

However, the assumption most likely to be seriously violated in a well-prepared test is that chance alone influences the response made by the

examinee to the multiple-choice item. For many examinees, some multiple-choice items are equivalent to the same item in answer-only form, since their wrong answer is among the presented options. Furthermore, partial credit may be given for partial knowledge since the examinee with some knowledge may be able to limit his selection to a choice among two or three options. Both of these factors would operate towards yielding scores in the multiple-choice form more nearly like those obtained in answer-only form and would hence contribute towards increased validity.

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A SECOND-ORDER FACTOR ANALYSIS OF REASONING ABILITIES*

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This article reports a second-order factor analysis of the 13 interpretable first-order factors of Adkins and Lyerly based on 66 variables for 200 Army men. The approach to simple structure for 6 second-order factors was unusually good. Loadings of the original 66 variables on the second-order factors were also obtained. Five of the factors are tentatively interpreted as Precision in Formation and Use of Verbal Concepts, General Verbal Fluency, Visualizing Spatial Constancy During Movement, Speed in Analysis, and Flexibility in Analysis.

In a factor analysis of the scores on 66 variables of 200 Army men, Adkins and Lyerly (**1**), attempting to clarify the nature of reasoning factors, obtained 13 interpretable oblique factors. This article reports the finding of 6 second-order factors from an analysis of the correlations between the 13 first-order factors.

The First-Order Factors

The 13 first-order factors, together with the tests that had loadings of .30 or over, presented in order of decreasing size, were as follows:

A′, *Perception of Abstract Similarities* (Verbal Classification II, Verbal Classification I, Figure Analogies, and Word Squares). This factor was described as the ability to perceive a type of similarity important in abstract thinking or as an ability to educe relations of similarity (**1**, 72–75).

B′, *Hypothesis Verification* (Progressive Matrices C, Progressive Matrices E, Verbal Analogies, False Premises). This factor was interpreted as the particular aspect of inductive thinking required in testing or verifying hypotheses (**1**, 76–80).

C′, *Verbal Relations* (Reading, Sentence Order, Vocabulary, Absurdities, Reading II, Practical Situations). The high loadings of Sentence Order and Absurdities make evident that this factor involves more than mere vocabulary, though the ability is undoubtedly identical to the factor commonly called "Verbal" (**1**, 64–65).

D′, *Perceptual Speed* (Forms, Block Counting, Progressive Matrices E, Identical Forms, Overlapping Circles). This factor agrees reasonably well

*This manuscript represents a condensation of a thesis presented by Mr. Matin for the degree of Master of Arts at The University of North Carolina, Miss Adkins having been chairman of the advisory committee.

with French's (**3**) description of Perceptual Speed, in which is stressed the task of finding quickly a given configuration in distracting material (**1**, 65–67).

F′, *Flexibility of Perceptual Closure* (Figure Classification IIA, Figure Classification IIB, Map Planning, Camouflaged Outlines, Identical Forms). Although this factor did not correspond fully with any previously identified factor, the best description was in terms of closure against distracting material (**1**, 80–85).

H′, *Deduction* (False Premises, Identical Forms, Word Squares). This factor was interpreted as the ability to draw correct inferences (**1**, 85–86).

J′, *Number* (Numerical Operations II, Numerical Operations I, Number Series, Arithmetic, Painted Blocks, Circles, Letter Series). This factor was identified as the familiar Number factor (**1**, 68).

K′, *Word Fluency* (Suffixes, First and Last Letters). The Word Fluency factor was described as the ability to produce quickly words that meet certain more or less mechanical specifications (**1**, 68–69).

L′, *Space 1* (Figures, Cards, Block Counting, Geometrical Puzzles). This factor corresponds closely with the factor that Thurstone calls Space 1, which seems to involve visualization of a rigid configuration when it is moved into different positions (**1**, 69–70).

M′, *Concept Formation* (Picture-Group Naming, Word-Group Naming, Verbal Analogies). The tentative hypothesis was that this factor represents the ability to formulate abstract or precise verbal concepts (**1**, 86–89).

N′, *Speed of Perceptual Closure* (Street Gestalt Completion, Identical Forms, Mutilated Words, Mutilated Pictures). This factor corresponds to Thurstone's Speed of Closure factor, interpreted as the ability to organize a chaotic visual field into a single percept (**1**, 70).

O′, *Ideational Fluency* (Topics, Things Round, Reasons). This factor appears to be the factor termed Ideational Fluency by other investigators and interpreted as the ability to produce ideas rapidly, regardless of their quality (**1**, 70–71).

P′, *Space 2* (Mechanical Information, Mechanical Movements, Practical Situations). This factor Adkins and Lyerly termed Space 2 (interpreted by Thurstone as the ability to visualize a configuration in which there is movement or displacement among its parts) only with some doubt, suggesting that the hypothesis that the variance in the factor was accounted for by "mechanical experience" had not been ruled out (**1**, 71–72).

The inter-correlations among the 13 first-order factors, which are reproduced below the main diagonal of Table 1, were obtained from the cosines of the first-order reference vectors (**1**, 122). By the complete centroid method (**5**, 161–170), six second-order factors were extracted, the factoring being repeated four times until communality values stabilized. The sixth-factor residuals are reproduced in Table 1 above the main diagonal. Projections of

TABLE 1*

Intercorrelation Coefficients of First-Order Factors (Below Diagonal) and Sixth-Factor Residuals (Above Diagonal)

	A'	B'	C'	D'	F'	H'	J'	K'	L'	M'	N'	O'	P'
A'		06	00	-01	00	08	-01	06	02	03	04	-01	00
B'	30		-02	01	-07	-03	-06	-09	-02	05	-07	-05	-01
C'	20	-13		-01	00	01	04	03	-04	-02	00	-03	-01
D'	16	10	-21		-02	-03	03	-04	-07	03	-03	01	00
F'	19	14	00	-20		08	04	-01	04	-02	-02	05	-06
H'	-11	-20	00	-13	-11		00	-01	-01	-05	-02	-06	-02
J'	48	06	22	34	08	-18		03	-01	-03	03	02	03
K'	50	02	18	-02	-12	-26	36		-02	-05	02	-02	00
L'	29	10	-04	06	34	09	07	-14		05	00	00	-04
M'	12	-06	38	-11	-14	-04	10	18	07		-06	-03	-04
N'	05	-05	-04	12	-19	-29	14	11	-05	-05		-03	-04
O'	10	-11	01	12	-39	-14	27	49	-41	18	05		-06
P'	04	08	-26	-09	30	26	-19	-19	43	-17	-23	-46	

* Decimal points have been omitted in all tables.

TABLE 2

Second-Order Centroid Factor Loadings

	I	II	III	IV	V	VI	h^2
A'	43	62	20	20	-06	14	67
B'	05	25	29	-06	-20	06	20
C'	30	33	-49	-16	38	12	62
D'	25	-14	52	23	14	10	44
F'	-38	56	11	-24	-19	17	59
H'	-42	-11	-30	45	14	22	54
J'	57	22	25	21	17	19	55
K'	63	26	-15	18	-37	-16	69
L'	-29	52	26	25	26	-15	57
M'	28	21	-40	07	23	-14	35
N'	26	-12	21	-14	10	-27	23
O'	63	-30	-22	26	-29	03	69
P'	-62	35	14	34	-12	-16	68

TABLE 3

Final Second-Order Transformation Matrix

	I	II	III	IV	V	VI
1	09	41	-22	30	-18	04
2	40	31	32	-27	-15	40
3	-72	-16	15	55	-35	05
4	-18	50	29	36	56	-21
5	33	-67	33	42	02	-20
6	-42	-10	-80	47	71	87

TABLE 4

Oblique Second-Order Factor Matrix

	I	II	III	IV	V	VI
A'	03	47	07	52	-03	36
B'	-19	15	-02	16	-14	22
C'	61	-04	-05	07	07	18
D'	-45	-01	01	52	00	-01
F'	02	-01	02	01	-07	45
H'	01	-04	02	00	60	-01
J'	-10	24	-06	61	04	21
K'	17	72	-03	01	-12	02
L'	09	-03	55	33	-08	-02
M'	51	13	15	-02	00	-10
N'	-01	-07	15	03	-37	-25
O'	-06	52	-32	-03	17	-08
P'	-05	10	46	-02	09	-07

TABLE 5

Second-Order Reference Vector Cosines

	I	II	III	IV	V	VI
II	00					
III	39	-02				
IV	-38	-03	-02			
V	-22	13	-46	25		
VI	-26	08	-69	39	41	

the first-order factors on the centroid reference vectors appear in Table 2. Rotation to oblique simple structure was accomplished by the single-plane method, with final adjustments by the radial method (**5**, 216–224, 194–216). The transformation matrix and the final oblique factor matrix are presented in Tables 3 and 4, respectively. It may be noted that the approach to simple structure, according to Thurstone's criteria (**5**, 335), appears to be unusually close for a second-order factorization. The reference vector cosines appear in Table 5. Although these cosines indicate appreciable correlation among some of the second-order factors, the authors are not inclined to suggest that a third-order analysis be performed, at least not until more is known about second-order analyses.

Some reviewers have expressed skepticism about determining as many as six factors from 13 variables. It may be noted, however, that as many as eight factors can be determined by 13 variables (**5**, p. 294). It should also be reported that the authors have explored the possibility that as few as four or five factors would be sufficient in this case, but that the residuals remained too large to be ignored. It is true, however, that one of the six factors, Factor *V*, has only one high positive loading and one fairly high negative loading. Although no interpretation is suggested for this factor, it can scarcely be dismissed as a residual factor.

Interpretations of Second-Order Factors

In the brief interpretations of the second-order factors, loadings exceeding .25 will be indicated. Some cognizance was also taken of the loadings of the original 66 variables on the second-order factors.

Although not a great deal of discussion will be devoted to the bearing of these original test loadings on the interpretation of the second-order factors, the complete table is presented in Table 6. No attempt was made to rotate the second-order factors to a satisfying fit in the test space. In general, interpretations of the second-order factors from inspection of the original test loadings are in essential agreement with those made from analysis of loadings of first-order factors. The loadings on second-order factors tend to hover around zero for tests that are significantly loaded on first-order factors that themselves have negative loadings on the second-order factors. In several instances, also, a test will have a possibly significant loading on one first-order factor that has a positive loading on a second-order factor, and also a possibly significant loading on another first-order factor that has a negative loading on the same second-order factor. The suggested interpretation of the latter finding is that the second-order bipolar factor represents two abilities, one at each pole, and that a subject may use either of the two alternative abilities in responding to such a test. Each bipolar factor has been named after the positive pole.

Factor I, Precision in Formation and Use of Verbal Concepts, has a loading

TABLE 6

Loadings of Tests on Second-Order Reference Vectors

Test	I	II	III	IV	V	VI
1. Absurdities	25	21	11	29	09	13
2. Arithmetic	13	17	20	39	10	15
3. Block Counting	-15	07	22	47	-05	13
4. Camouflaged Outlines	04	28	17	28	-09	21
5. Cards	07	11	41	34	-04	04
6. Circles	-06	20	04	40	18	26
7. Conclusions	05	32	-03	20	00	15
8. Decoding	15	36	10	38	08	30
9. Designs	03	23	26	34	-08	12
10. False Premises	-03	16	07	25	25	14
11. Figure Analogies	05	38	19	39	06	26
12. Fig. Class. I	-00	28	11	41	00	22
13. Fig. Class. IIA	05	24	10	32	-07	37
14. Fig. Class. IIB	-07	33	02	33	02	40
15. Figure Series	04	20	11	45	05	21
16. Figures	10	12	34	33	-06	05
17. First and Last Letters	04	55	04	10	-03	06
18. Forms	03	16	17	32	09	-00
19. Geometrical Puzzles	05	22	21	36	01	18
20. Identical Forms	-05	16	08	40	-01	13
21. Incomplete Outlines	09	39	08	50	11	27
22. Letter Series	09	24	09	44	-01	24
23. Logical Puzzles	21	28	11	27	07	18
24. Map Planning	07	28	09	26	01	19
25. Matrices VI	11	28	24	46	-09	17
26. Mechanical Information	04	14	26	05	06	05
27. Mechanical Movements	11	26	30	08	-01	04
28. Mixed Series	16	41	12	46	03	26
29. Mutilated Pictures	-04	15	14	32	-11	02
30. Mutilated Words	20	23	23	14	-19	-11
31. Nim	02	10	13	24	-04	08
32. Number Series	00	18	11	46	08	26
33. Numerical Operations I	-05	28	00	48	08	26
34. Numerical Operations II	05	29	02	53	12	24
35. Numerical Puzzles	09	19	12	10	10	02
36. Overlapping Circles	-07	15	12	41	05	17
37. Painted Blocks	01	18	16	36	06	08
38. Paper Folding	-04	19	34	42	04	13
39. Picture Analogies	02	27	10	43	-02	22
40. Picture Arrangement	14	05	15	46	-00	09
41. Picture Classification	05	00	12	28	-07	07
42. Picture-Group Naming	31	41	17	20	-11	-05
43. Planning A Circuit	-08	11	30	30	.01	05
44. Practical Situations	24	34	13	08	-02	13
45. Progressive Matrices B	-04	21	13	32	-02	20
46. Progressive Matrices C	-03	30	09	32	-12	22
47. Progressive Matrices D	-02	18	08	32	-05	26
48. Progressive Matrices E	-04	22	23	40	-01	11
49. Reading	41	38	11	27	03	17
50. Reading II	28	29	05	30	20	22
51. Reasons	-01	44	-15	11	08	01
52. Sentence Order	34	27	-01	30	16	21
53. Series	03	39	17	36	12	24
54. St. Gestalt Completion	28	37	16	12	-12	-09
55. Suffixes	23	57	10	07	-14	-00
56. Surface Development	-08	14	34	40	-14	07
57. Things Round	08	52	-12	12	12	07
58. Topics	-06	40	-04	14	00	-11
59. Verbal Analogies	24	42	10	34	14	29
60. Verbal Class. I	17	28	06	50	01	34
61. Verbal Class. II	16	41	06	45	02	34
62. Vocabulary	34	45	16	30	-03	15
63. Word-Group Naming	23	45	08	36	04	14
64. Word Selection	20	36	21	31	03	25
65. Word Squares	05	34	08	49	18	29
66. Education	10	48	-02	33	02	18

of .61 for C′, Verbal Relations; .51 for M′, Concept Formation; and −.45 for D′, Perceptual Speed. This factor seems to involve the ability to form and use verbal concepts precisely. The tests of both Verbal Relations and Concept Formation involve the analytic manipulation or the eduction of verbal concepts. Thus the fact that several tests of Verbal Relations did not depend upon high-level vocabulary suggested an interpretation stressing the perception and manipulation of verbal relations rather than merely understanding the meaning of words.

Here, as in the other bipolar second-order factors to be reported, those tests with significant loadings on the first-order factor which is at the negative pole have near-zero loadings on the second-order factor. Both Forms and Progressive Matrices E, which have their highest loadings on Perceptual Speed, also have loadings slightly above .25 on Concept Formation, which is negatively correlated with Perceptual Speed (−.11). The interpretation may be that the two poles involve alternative or even antagonistic processes, either of which is used by some subjects in the performance of these tests.

Since all the tests with loadings on this second-order factor greater than .25 (except Street Gestalt Completion, with a loading of .28) also have significant and positive loadings on either Verbal Relations or Concept Formation, the interpretation of the second-order factor based on loadings of the first-order factors at the positive pole is substantiated by consideration of loadings of the original tests on the second-order factor.

Factor II, General Verbal Fluency, has a loading of .72 for Factor K′, Word Fluency; .52 for O′, Ideational Fluency; and .47 for A′, Perception of Abstract Similarities.

The high loadings on this factor of Word Fluency and Ideational Fluency suggest a more generalized fluency ability. In the tests that were loaded on Perception of Abstract Similarities, too, a type of fluency in educing possible relations to be tested for similarity seems to be essential. The restriction of the task set for the subject is apparently least for the tests of the first-order factor with the highest loading on the second-order factor, Word Fluency, and greatest for the one with lowest significant loading, Perception of Abstract Similarities, thus permitting greater involvement of a fluency factor in the tests of Word Fluency than in the tests of Perception of Abstract Similarities.

A large number of the original tests have loadings above .25 on this general fluency factor. Since nearly all with loadings above .40 very clearly entail the use of words, perhaps the factor should be termed General Verbal Fluency.

Factor III, Visualizing Spatial Constancy During Movement, has a loading of .55 for Factor L′, Space 1; .46 for P′, Space 2; and − .32 for O′, Ideational Fluency. If the interpretation of Space 2 is tenable, the best hypothesis for this factor is that it calls for visualization involving maintenance of a framework containing parts that can move conceptually while, at the same

time, certain relations among the parts remain constant. In the reference tests for Space 1, rigid configurations are visualized as moving against a homogeneous background. In Mechanical Movements, which is highly loaded on Space 2, the "framework" is part of the object against which other parts move in relation to each other. Here the subject must maintain a conceptual object constancy while visualizing movement.

Although there is no readily apparent reason for the negative loading of Ideational Fluency on this factor, it may have arisen from the loadings for Topics of .25 on Space 1 and of .52 on Ideational Fluency, which correlated −.41 with Space 1. Again, then, it is suggested that a bipolar second-order factor may mean simply that certain tests can be solved by alternative or antagonistic abilities.

Factor IV, Speed in Analysis, has a loading of .61 on J′, Number; .61 on A′, Perception of Abstract Similarities; .52 on D′, Perceptual Speed; and .33 on L′, Space 1.

The high loadings on Number and Perceptual Speed immediately suggest some type of speed factor, apparently quite general in character as indicated by the diversity of the four first-order factors. Speed, however, may be simply an indicator of a high level of a given ability rather than a separate ability in itself. On this basis, some more fundamental ability common to these factors is suggested. The most plausible suggestion concerns analytical ability, possibly at the perceptual level. Perception of Abstract Similarities and Perceptual Speed are obviously analytical in nature. Space 1 requires the separation of parts of a perceptual gestalt. Some of the tests of Number, the two series tests, and Circles also call for analytical ability. While analysis does not seem to be necessary for simple arithmetic computations, analytic ability may be important in the original learning of arithmetical operations.

Twenty-eight of the 30 tests with loadings above .25 on one of the four first-order factors that have high loadings on the second-order Factor IV also have loadings above .25 on it. Twenty-six additional tests, or 54 in all, have loadings above .25 on Factor IV. Such a large number of highly loaded tests gives the factor the appearance of a general factor; yet such an interpretation is scarcely consistent with the loadings of the first-order factors alone. Pending further clarification, it seems preferable to restrict the interpretation to that based on the loadings of the first-order factors, which resulted from rotation to simple structure.

Factor V has a loading of .60 for H′, the first-order Deduction factor, and −.37 for N′, Speed of Perceptual Closure. Since only one positive loading, that for Deduction, exceeds .25, no attempt will be made to name this factor. The deductive factor was not over-determined in the first-order analysis, a fact which weakens the significance of findings relating to the second-order factor.

The Identical Forms test, with a loading of .31 on Deduction and .48

on Speed of Perceptual Closure, may account for the negative loading of the latter first-order factor on Factor V. Again, the two first-order factors concerned are negatively correlated, −.29. As before, probably some subjects use one of the abilities and some the other.

Factor VI, Flexibility in Analysis, has a loading of .45 for F′, Flexibility of Perceptual Closure; .36 for A′, Perception of Abstract Similarities; and −.25 for N′, Speed of Perceptual Closure.

The positively-loaded first-order factors suggest that Factor VI involves analytic ability in some manner. The inductive nature of Perception of Abstract Similarities was discussed in the report on the first-order analysis (**1**, 73). Other recent studies have shown a relation between the Inductive and Flexibility of Closure factors (**2**, 4).

The tests of Perception of Abstract Similarities and those of Flexibility of Perceptual Closure resemble each other. Nearly all the classification and analogies tests in the original battery are loaded on one or the other of them. The analytic nature of the abilities required by these tests is apparent, both factors involving the finding of common elements in distracting materials.

In Botzum's study, Flexibility of Closure and Speed of Closure were both positively loaded on the same second-order factor (**2**). Pemberton found the two almost orthogonal and not appearing on the same second-order factor (**4**). These findings are neither consistent with each other nor with the findings of the present study, in which the two closure factors correlate −.19 and appear, one with a positive loading and one with a negative loading of borderline significance, on the same second-order factor. The apparent bipolarity that was found in this study probably resulted from the fact that Identical Forms and Camouflaged Outlines had loadings slightly greater than .25 on both factors, which themselves were negatively related. It should be noted that since the identification of the Flexibility of Closure factor in the first-order analysis was not very certain, the interpretation of this second-order factor is still highly tentative.

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NOTE ON THE SELECTION OF A PANEL OF JUDGES SO AS TO MAXIMIZE PANEL EFFICIENCY*

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Attempts to develop improved methods of selecting a panel of judges for psychometric work are presented. The applications reported are in the sensory field and are limited to the assumption of a unidimensional ability underlying the judgments in question. Some aspects of cost efficiency are also considered.

Introduction

The situation is quite frequently encountered in psychometric work in which a select panel of judges is to be formed from a pool of available individuals. The object is to make the selection in such a way as to maximize the power of the panel in making statistically reliable discriminations. These judgments or discriminations may be in any of a number of areas in the psychophysical or psychometric field—e.g., taste testing, color discrimination, and so on. Selection of judges is based upon scores which may have been derived either from past routine performance or from a special series of screening tests.

Hitherto, very arbitrary standards of selection seem to have been used. E.g., it may be decided to select the best 50 per cent of the group, or to select those attaining, say, 3/4 of the maximum score in the screening test. The present note describes some attempts to develop a more rational approach to this problem, and it is hoped that others will be stimulated to carry the argument forward at a more rigorous level.

The applications reported here are in the sensory field (taste testing), but the conclusions presumably can be generalized to cover other psychophysical and psychometric fields. Only the simplest case will be considered, where it can be assumed that the basis of judgment is a unidimensional ability. Furthermore, this application will assume that each judge can make only one judgment on each item. This restriction is dictated by the rapid fatigue associated with taste-test work and by the necessity of not impairing motivation of the tasters.

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Some aspects of cost efficiency will also be considered, where the problem is to form a select panel which will return the best value for each operating dollar expended. In addition, an interesting simplified model will be investigated, in which the abilities of the pool of judges from which selection is to be made are linearly decreasing when ranked in diminishing order of magnitude. A frequent observation has been that the corrected screening scores of candidate tasters do fall approximately on a straight line when so ranked.

The Problem

Suppose that there are from 100 to 200 persons from among whom a taste-test panel is to be formed. Then the problem is to select those tasters, based on screening scores, who will maximize the efficiency of the panel so formed.

The Screening Procedure

Some of the details described in this section are peculiar to the taste-testing problem and not of general application. Their inclusion is necessary to make the example clear.

Panels must be recruited separately for each food product; this does not, of course, prevent a given individual from serving on several panels, provided he can qualify. A heterogeneous series of samples is assembled which seem to cover as well as possible the full gamut of flavor nuances to be expected under routine conditions. The samples are presented in groups of three; in each such group two of the samples are identical. The task of the subject is to identify the odd sample. All samples are, of course, suitably coded to conceal the identity of the odd sample. This "triangular" procedure, as it has come to be called, is associated with a chance expectancy of 1/3 and is equivalent to a multiple-choice question with three alternative answers. A large number of such triangular sets are administered over a period of several weeks. For details of administration, the reader is referred to Bengtsson and Helm (**1**), Harrison and Elder (**8**), and Peryam and Swartz (**9**).

This procedure ignores the whole question of flavor-dimensionality. Presumably a more rigorous approach to the problem would be through the use of such mathematical tools as multiple-factor analysis, item analysis, and discriminant functions (**2**, **3**, **6**, **7**, **10**), in order to establish the flavor-dimensionality of a given food product. This would involve more extensive research than the authors have been prepared to undertake, especially as it is far from clear that this would lead to any large gain in precision in selecting tasters. The screening method used, then, gives a single measure of ability for each taster, without giving any indication of how his ability is distributed among the various flavor areas. This restriction somewhat modifies the conclusion about the number of persons who ought to be selected for optimum

panel efficiency. No attempt was made to correct scores for item-difficulty, since there seems to be serious doubt as to whether the increase in precision so obtained is sufficient to justify the computational labor. (**4**, **5**).

Treatment of Data

The percentage of times the odd sample was correctly selected is recorded for each judge, this figure representing the raw score of his ability. These scores are corrected so as to eliminate the expected number of correct selections due to chance in each case, using the simple formula derived from Guilford (**4**, 445),

$$S = \frac{100(R - C)}{100 - C},$$

where S = percentage score corrected for chance expectation,

R = raw percentage score, and

C = percentage score expected by chance.

In this case, of course, $C = 33\ 1/3$ per cent. This correction amounts to a simple linear transformation which in no way alters the relative scores. An example of the corrected screening scores of 15 participants is shown in Table 1.

TABLE 1

Tasters' Scores Ranked in Decreasing Order of Magnitude, After Correction for Chance Expectancy

Rank	Score
1	70%
2	64%
3	59%
4	53%
5	47%
6	47%
7	37%
8	29%
9	21%
10	14%
11	14%
12	10%
13	7%
14	(−1%)
15	(−5%)

Efficiency of a panel is defined in terms of the probability of that panel's detecting significant differences between samples presented to it. This will clearly be a function of two variables, namely, the mean ability of the panel and the available degrees of freedom.

Let the number of tasters who are eventually selected from the N individuals given the screening tests be designated by n. Now imagine, for simplicity, that subsequent routine testing employing this select panel will be of some non-parametric variety such as the triangle tests upon which the individuals were screened. Then the efficiency of the select panel in the sense defined may be measured by the size of the critical ratio obtained in any given instance. This critical ratio will be given by

$$\text{Efficiency} = \text{Critical Ratio} = \frac{R' - C}{\sqrt{\dfrac{C(100 - C)}{n}}},$$

where n is the size of the panel and R' is the raw percentage score obtained by this panel. We now make two simplifying assumptions:

(1) That the ability of a taster to detect differences is proportional to his corrected screening score. The mean ability of the n selected persons will then be given by

$$\sum S_i/n.$$

Since the highest corrected screening score was seldom above 70 per cent, it was felt that this assumption would not lead to large distortions.

(2) That the observed difference $(R' - C)$ in any given subsequent routine test would be proportional to the mean ability of the panel, i.e.,

$$R' - C = k \sum_{i=1}^{n} S_i/n,$$

where k is a suitable constant. Hence:

$$\text{Efficiency} = \frac{k \sum\limits_{i=1}^{n} S_i/n}{\sqrt{\dfrac{C(100 - C)}{n}}} = \frac{k}{\sqrt{C(100 - C)}} \cdot \frac{\sum\limits_{i=1}^{n} S_i}{\sqrt{n}} .$$

It is assumed that our tasters are ranked in order of screening score, so that S_1 corresponds to the man with the highest score. It can now be determined how the efficiency is related to the select panel size n. Since in any one instance, the term $k\sqrt{C(100 - C)}$ will be constant, it may be dropped without altering the relative efficiency obtained. Using the example set forth in Table 1, then,

(1) If a one-man panel is formed with the best individual,

$$\text{Efficiency } (E) = 70/\sqrt{1} = 70;$$

(2) Taking the two best men,

$$E = (70 + 64)/\sqrt{2} = 91.5.$$

This procedure was repeated until all N of the original persons screened were included. The upper curve in Figure 1 shows the relation between panel size and efficiency. It will be seen that the efficiency passes through a maximum where $n = 8$. It would, therefore, appear that selecting those persons with the eight top screening scores will yield a select panel with maximum efficiency.

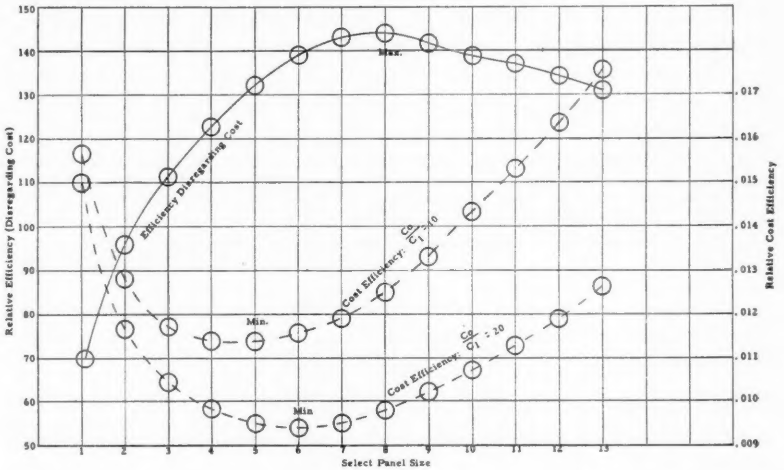

FIGURE 1
Relationship of Relative Efficiency to Size of Select Panel

The reliability of this estimate of the maximum value is limited by the reliability of the original scores, particularly the top taster's score. All scores should therefore be based upon large numbers of judgments so that they are reasonably reliable.

There are several reasons why it might be desired to take one or two *additional* persons into our select panel:

(1) Typically, efficiency diminishes more rapidly to the left of the maximum than to the right; it is presumably safer, therefore, to err on the side of having too many tasters than too few.

(2) It has been assumed, in effect, that the taste ability of our judges is unidimensional, being represented by their over-all screening scores. As pointed out above, this will certainly not be the case. Consequently, added tasters may contribute more to the efficiency than the approach has indicated just because tasters may tend to complement one another in their abilities. This again would suggest that the derived maximum value may be a little too small.

(3) It must be pointed out that the dispersion of screening scores obtained does not provide an unbiased estimate of the corresponding population dispersion, owing to the operation of sampling errors. These have the effect of inflating the observed screening scores of the top half of the group and depressing the scores of the lower half. The seriousness of this effect depends (inversely) upon the number of triangular items included in the screening test. Again, the effect will be to cause the number of tasters giving maximum panel efficiency to be somewhat too low, since sampling errors are causing the men with the higher scores to appear better than they probably are.

The authors have repeatedly observed that a set of ranked screening scores, represented graphically as a bar diagram, form an approximately linear "staircase." Although no reason why a set of scores should conform to such a simple rational function was apparent, it was decided to investigate the dependence of efficiency upon n if it were assumed that the scores ranked in order of magnitude do fall on a straight line.

$\sum S_i$ can now be replaced by an arithmetical progression,

$$\frac{n}{2}\{2S_1 - (n - 1)d\},$$

where

S_1 is the highest screening score obtained in the group and

d is the common (negative) difference in score between any two contiguous persons.

We then have

$$E = \frac{k}{2\sqrt{pq}} \cdot \frac{n\{2S_1 - (n - 1)d\}}{\sqrt{n}} .$$

Taking the first derivative of E with respect to n, we have

$$\frac{dE}{dn} = \frac{k}{2\sqrt{pq}} \left\{\left(S_1 + \frac{d}{2}\right)n^{-\frac{1}{2}} - \frac{3}{2}\, dn^{\frac{1}{2}}\right\}.$$

Equating to zero and solving for n, we find

$$\left(S_1 + \frac{d}{2}\right)n^{-\frac{1}{2}} - \frac{3}{2}\, dn^{\frac{1}{2}} = 0,$$

$$n = \frac{1}{3}\left(\frac{2S_1}{d} + 1\right).$$

The nth person in such a selected group will clearly have a screening score of $S_n = S_1 - (n - 1)d$. Expressed as a fraction of the best taster's score,

this will be

$$\frac{S_n}{S_1} = \frac{S_1 - (n-1)d}{S_1}\,.$$

Substituting for n, we have

$$\frac{S_n}{S_1} = \frac{S_1 - \left[\frac{1}{3}\left(\frac{2S_1}{d} + 1\right) - 1\right]d}{S_1} = \frac{1}{3} + \frac{2}{3}\frac{d}{S_1}\,.$$

The value of this conclusion is critically dependent upon the reliability of the tasters' scores. It is therefore desirable for this and other reasons that these be reasonably reliable, i.e., be based upon a large number of judgments. As the number of persons participating in the screening test becomes large, the above value approximates 1/3. The important conclusion follows that *where the ranked screening scores of a group of persons form an approximately linear "staircase" the most efficient panel is obtained by selecting all persons scoring* 1/3 *or more of the top person's score.*

Cost Efficiency

If it is desired to take the cost of running a taste-test panel into account in arriving at an estimate of the number of tasters to select to maximize the panel efficiency, the conclusion is altered somewhat. Taste-panel costing depends upon two elements:

C_0 = fixed cost of setting up a taste test, independent (up to a point) of the number of tasters participating, and
C_1 = additional cost per taster.

For a panel of size n, the total cost will therefore be given by

$$C_0 + C_1 n.$$

The cost efficiency may be obtained by dividing this cost by the efficiency estimate previously obtained above. We therefore have

$$\text{Cost Efficiency} = \frac{C_0 + nC_1}{\frac{k}{\sqrt{pq}} \cdot \sum_{i=1}^{n} S_i/\sqrt{n}}\,.$$

As before, $k/\sqrt{pq}$ in any given instance will be a constant multiplier independent of n, so

$$\text{Cost Efficiency } (E_c) = \frac{C_0 + nC_1}{\sum_{i=1}^{n} S_i/\sqrt{n}}.$$

The panel efficiency for the data given in Table 1 is shown plotted against n in Figure 1. The lower two curves in that figure correspond to two different

cost-situations, one in which $C_0/C_1 = 10$ and the other where $C_0/C_1 = 20$. C_0 has been put equal to unity. Comparing the two measures of efficiency discussed, the one ignoring cost and the other taking cost into account, it may be noted that

(1) Taking cost into account reduces the panel size. This is to be expected since each additional taster included in the select panel costs money.

(2) When $C_1 = 0$, then the cost efficiency merely becomes the inverse of the efficiency disregarding cost.

(3) When $C_0 = 0$,

$$E_c = \frac{n\sqrt{n}}{\sum_{i=1}^{n} S_i} C_1 .$$

In other words, cost rises so sharply that in all cases the most efficient panel would be a one-man panel! But in order to obtain statistically significant results, this man would have to make a number of repeat judgments. In routine testing work, the authors generally have been reluctant to ask tasters to judge more than once or twice a day, so that in this situation it would be necessary to increase panel size in the face of less economical operation.

When tasters' ranked screening scores form an approximately linear "staircase" the rational function for cost efficiency is given by

$$\text{Cost Efficiency} = E_c = \frac{C_0 + nC_1}{\dfrac{k}{2\sqrt{pq}} \sqrt{n}\,\{2S_1 - (n-1)d\}} .$$

Proceeding as before, differentiating with respect to n, equating to zero, and solving for n, we have

$$\frac{d(E_c)}{dn} = \frac{\dfrac{k}{2\sqrt{pq}} [(2S_1 + d)n^{\frac{1}{2}} - dn^{\frac{3}{2}}]C_1}{\left[\dfrac{k}{2\sqrt{pq}} \sqrt{n}\,\{2S_1 - (n-1)d\}\right]^2} - \frac{\dfrac{k}{2\sqrt{pq}} [\frac{1}{2}(2S_1 + d)n^{-\frac{1}{2}} - \frac{3}{2}\, dn^{\frac{1}{2}}](C_0 + nC_1)}{\left[\dfrac{k}{2\sqrt{pq}} \sqrt{n}\,\{2S_1 - (n-1)d\}\right]^2}$$

$$= C_1\, dn^2 + \{C_1(2S_1 + d) + 3C_0 d\}n - C_0(2S_1 + d) = 0.$$

The roots of this quadratic are given by

$$n = \frac{-\left\{\left(\dfrac{2S_1}{d} + 1\right) + 3\dfrac{C_0}{C_1}\right\} \pm \sqrt{\left(\dfrac{2S_1}{d} + 1\right)^2 + 10\left(\dfrac{2S_1}{d} + 1\right)\dfrac{C_0}{C_1} + 9\left(\dfrac{C_0}{C_1}\right)^2}}{2} .$$

Only the positive root is meaningful. It will be seen that the value of n depends upon two quantities only, C_0/C_1 and $(2S_1/d + 1)$. The first quantity is therefore simply the ratio of the two component costs, and the second quantity may be considered equal to $2N$, where N is a continuous measure of the limiting number of tasters available for a select panel; tasters scoring below chance expectancy in the screening test are assumed to be excluded. So, we can put

$$n = \sqrt{N^2 + 5N\frac{C_0}{C_1} + \frac{9}{4}\left(\frac{C_0}{C_1}\right)^2} - \left(N + \frac{3}{2}\frac{C_0}{C_1}\right).$$

Now, where S_n = screening score of the nth taster,

$$S_n = S_1 - (n - 1)d,$$

and

$$\frac{S_n}{S_1} = \frac{(N - \frac{1}{2})d - (n - 1)d}{(N - \frac{1}{2})d} = \frac{N - n + \frac{1}{2}}{N - \frac{1}{2}}.$$

Finally, substituting in this equation for n, we get

$$\frac{S_n}{S_1} = \frac{2N - \sqrt{N^2 + 5N\frac{C_0}{C_1} + \frac{9}{4}\left(\frac{C_0}{C_1}\right)^2} + \frac{3}{2}\frac{C_0}{C_1} + \frac{1}{2}}{N - \frac{1}{2}}.$$

This fraction marks the dividing point between tasters who should be included in, or excluded from, our select panel. Tasters having screening scores above this fraction of the best taster's screening score qualify for inclusion, whereas those scoring below should be excluded, if the most cost-efficient select panel is to be formed.

A series of different values for (N) and C_0/C_1 were substituted in the above equation and the value of S_n/S_1 calculated in each case, in order to draw some heuristic conclusions about the dependence of S_n/S_1 upon these two parameters.

The resulting family of curves is shown in Figure 2. It can be seen, for example, that when $N = 20$ to 50, and C_0/C_1 is in the region of 10 to 20, $S_n/S_1 = 3/4$; i.e., in these circumstances all tasters scoring 3/4 or more of the top screening score should be selected for the panel in order to maximize cost efficiency. Should these persons be too few to give reasonable sensitivity, the panel size may be increased, but not beyond the point where $S_n/S_1 = 1/3$, since this would give the most efficient panel irrespective of cost.

It must be pointed out that the solution to the maximum problem treated in this note would have been much simpler but for a restriction imposed by the nature of the psychophysical task. This restriction is that in taste-test work under routine conditions, it is not feasible to have each taster evaluate a group of samples more than once. Any attempt to make him do so regularly

results in fatigue and loss of motivation. If it were not for this restriction, then obviously the most efficient evaluation procedure would be to select the best man and subject him to a full-fledged psychophysical session with

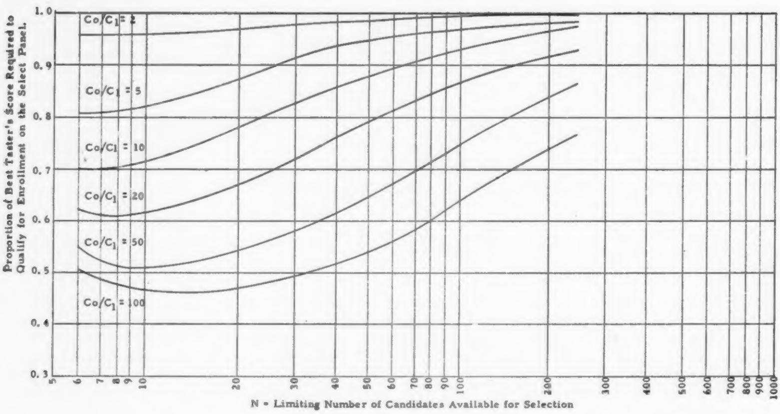

FIGURE 2
Panel Selection Maximizing Cost Efficiency for Different Cost Ratios

the same group of samples presented a sufficient number of times to secure the desired degree of sensitivity and statistical assurance. Such might be the case in the fields of vision and audition.

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BOOK REVIEWS

CLYDE H. COOMBS. *A Theory of Psychological Scaling.* Ann Arbor: University of Michigan Press, 1952, pp. vi + 94.

At the outset of this monograph, Coombs expresses his dissatisfaction with what might be called the traditional and current theory of psychological measurement. It is characteristic of this theory, Coombs argues, to build strong mathematical models. Furthermore, the model builders then proceed to make assumptions about behavior that will adapt the behavior to the mathematical model. If the observations of behavior (data) fail to fit the model, then the statistical notions of error theory, random variation, and so forth, are used to account for or explain away the discrepancies. It is the data that suffer, not the theory or mathematical model.

Coombs, on the other hand, believes that ". . . the integrity of the data [should] be maintained and the level of measurement adapted to the information in the data" (p. 9). Since he also believes that the information contained in the data is very crude, he contends that only weak postulates can be expected to be satisfied by the data. It is necessary, therefore, to build mathematical models based upon weak postulates if behavior is to be described adequately and without resort to the notions of error theory and random variation. "To impose stronger systems in spite of data is to build an actuarial science at the possible cost of a science of individual behavior" (p. 9).

This is the basic issue, as Coombs sees it, between the theory of scaling that he develops and the traditional theory of psychological measurement. A further contrast, however, can be made between the two approaches. Psychometric theory, he argues, has had as its objective the *construction* of scales. It has, therefore, been much concerned over the development of techniques and methods which would offer some assurance of success in the task of constructing, preferably, unidimensional scales. In this objective Coombs is not at all interested. Rather he sets as his objective the *discovery* of psychological scales. If a scale exists for a set of stimuli, this fact alone is of primary psychological significance. But, he points out, we cannot conclude that a scale exists if the technique used is such as to guarantee its discovery.

Chapter 1 of the monograph discusses briefly measurement theory both from the formal or logical side and from the observational or experimental side. To S. S. Stevens' description of the ratio, interval, ordinal, and nominal scales, Coombs adds the ordered metric and the partially ordered scale. The former is one "in which the order of the objects on a continuum is known, and also the distances between objects may themselves be at least partially ordered" (p. 3). No precise statement was found defining the partially ordered scale, but it would appear from context that by this concept Coombs means that some of the stimuli can be ordered on a continuum, but not all. (It is unfortunate that this 94-page monograph has no index.) Taking first the scale that holds when the stimuli are regarded as elements and then the scale that holds when the distances between the stimuli are regarded as elements, Coombs develops a more complete listing of possible types of scales.

A distinction is made between psychological measurement and psychological scaling, although Coombs believes that in the most general sense all scales involve measurement as a matter of degree. If the operations of arithmetic are permissible, so that numbers can be assigned to the objects or their differences, then measurement is involved. If these operations are not permissible, then scaling is involved. "On this basis, ratio and interval scales are both classified as measurement, and the remaining scales, including the ordered metric, ordinal scale, partially ordered scale, and nominal scale are classified as scaling" (p. 5).

Since Coombs has only a weak postulational system (no postulates which lead to measurement are included within the system), his interest is in these latter scales.

Chapter 2 is concerned with the problem of psychological measurement and scaling. Coombs makes the distinction, after Lewin and others, between phenotypic and genotypic systems. The genotypic system refers to the underlying, hypothetical basis of behavior. The phenotypic system refers to the observed behavior. This distinction is similar to that made by Lazarsfeld in his latent attribute analysis, in which the latent attribute is called upon to explain manifest behavior. Both the approach of Coombs and that of Lazarsfeld stand in marked contrast to Guttman's scale analysis, which is concerned only with the manifest behavior. For Guttman an attitude is a sample of behavior. For Coombs and Lazarsfeld an attitude would be "that" which underlies the behavior.

The postulation of a latent or genotypic "that" raises interesting philosophical and methodological questions which cannot be discussed in detail here. This reviewer, for one, however, has grave doubts about the usefulness of hypothetical entities that are called upon to explain behavior, whether they are called drives, forces, latent attributes, hypothetical constructs, or genotypic systems—if these entities themselves cannot be defined in terms of observations or operations upon observations. Coombs, on the other hand, although concerned with weak postulational systems that will adequately describe behavior, also states that his objective is to gain, from the information contained in the phenotypic observations, inferences about the underlying genotypic system. The genotypic system, in turn, it may be noted, is to be used to explain the phenotypic observations.

In Chapter 3, Coombs gives his definitions and postulates. Here we find the distinction made between Task A and Task B. Task A is concerned with the preference judgments of stimuli with the individual as point of reference. Task B is concerned with judgments of stimuli with respect to some attribute. For Task B, the individual's own position on the continuum under investigation is irrelevant, i.e., the judgments made are assumed to be independent of the location of the judge on the continuum. This is Thurstone's contention, stated in 1939, that judgments of the degree of favorableness and unfavorableness of attitude items are independent of the attitudes of the subjects doing the judging.

The researches of Hinckley and others on this point, in the thirties, seemed conclusive in showing that Task B judgments were, in fact, independent of the attitudes of the judges. Still, some of us were concerned about the issue. For example, in 1946, this reviewer and K. C. Kenney stated:

> We are not satisfied with the evidence on this point. Would similar results obtain from judgments derived from those with sympathetic attitudes toward fascism and those violently opposed to fascism in the construction of a scale measuring attitude toward fascism? And in the case of communist sympathizers and non-communists in the construction of a scale measuring attitude toward communism? When social approval or disapproval attaches to a favorable or unfavorable attitude toward an issue, different scale values might result from groups with differing attitudes. . . . The research so far, it seems to us, also neglects the related problem of *ego-involved* attitudes and the bearing they might have upon scale values of items.

Now we find that recent research by Hovland and Sherif provides evidence that the procedures used in scaling attitude items by the method of equal-appearing intervals ruled out the possibility of the data showing the influence of the judges' attitudes upon the scale values obtained. The reason was a simple one, namely, that individuals with extreme attitudes were incidentally eliminated by a criterion of "carelessness." Hovland and Sherif show that it is precisely these "careless" judges who fall at the two extreme ends of the continuum. And the scale values of items obtained from the judgments of these extreme groups do show the influence of the judges' positions on the continuum.

Coombs will have to reconcile the findings of Hovland and Sherif with the assumption

he has made concerning the irrelevance of the position of the individual for judgments obtained in Task B. Since he makes no pretension of providing a completed theory (there are many departures from the position taken in his earlier articles), the necessary modifications in his postulational system should be possible.

The first three chapters, described briefly above, contain the essentials of Coombs' theory. Chapter 4 develops equations for the genotypic parameters and Chapter 5 for the phenotypic parameters. Chapter 6 then deals with certain interrelations between the genotypic and phenotypic parameters.

Chapter 7 is concerned with the "unfolding technique," a method which Coombs believes enables one "to go behind the expressed preferences of individuals and to construct a model from which their preferences may be *derived*" (p. 56). In this chapter we also find the statement that "there is no objective way in the Method of Paired Comparisons of testing the consistency of an individual's judgments" (p. 71). In an earlier article (1950), Coombs had defined consistency in terms of the phenotypic system. In the present monograph, he defined consistency in terms of the genotypic system. If consistency refers to the underlying genotypic system, Coombs' statement regarding a test for consistency might be accepted. However, on the phenotypic level, Kendall's work on paired comparisons, his development of a coefficient of consistence, and the χ^2 test of significance of the coefficient of consistence would seem applicable.

The monograph concludes with Chapter 8, which is concerned with some of the implications of the scaling theory developed for Thurstone's law of comparative judgment. Coombs believes that properly there are two such laws, one for Task A and one for Task B; i.e., judgments of preference and judgments of attributes, respectively. Again some modification of this section will apparently be necessary to take into account the empirical findings of Hovland and Sherif.

An over-all judgment of Coombs' contribution would be an exceedingly difficult task, regardless of whether the judgment is made from the point of view of Task A or Task B. Without concern for the nature of the scale involved, the Task B judgment of the reviewer is that in some respects Coombs falls somewhere near Lazarsfeld, in that both, for example, are interested in genotypic or latent systems. In some respects he also falls near Guttman, for both are interested in the discovery rather than the construction of scales. In terms of techniques used for the collection of data, such as the method of paired comparisons, Coombs might be placed near Thurstone.

Again without concern for the nature of the scale involved, from a Task A point of view, the judgment is easier. From a preference point of view, this reviewer belongs at that point on a continuum where fall those who like strong postulational systems, the consequent operations of arithmetic, and the applications of error theory and random variation. Coombs, by choice, has scaled himself at the point where those who prefer weak postulational systems fall. Consequently, if a single continuum is involved, Coombs is at one end and the reviewer is at the other.

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KENNETH L. BEAN. *Construction of Educational and Personnel Tests*. New York: McGraw-Hill, 1953, pp. viii + 231.

The author begins with the premise that the majority of teachers and others who are faced with the job of constructing and using tests have not had sufficient training in test construction. This condition is attributed essentially to inadequate curricula and to a lack of textual treatises on the subject of test construction.

It is specifically stated that "Most individuals who are interested in using test

results, except research workers, do not like to become lost in technical verbiage beyond their comprehension." Accordingly, the book is written in a style and manner which may not appeal to the highly qualified psychologist but which will be appreciated by the greater portion of test constructors.

The book is not a thorough treatise of the subject of test construction; however, it does present many of the essentials of test construction in such a manner that potential test constructors with a moderate amount of sophistication will benefit from reading it. On the more technical aspects of test contruction, the author makes ample reference to sources of information which are of a higher technical level.

The author presents elementary concepts of measurement in a very understandable manner. There is a good discussion of terminology in an attempt to clarify some of the related terms such as aptitude, capacity, and talent. There is coverage of such matters as the distinction between achievement and aptitude measurement.

The author fully recognizes the importance of the often neglected steps of planning the examination. He recognizes the importance of job analysis and planning for the administration and scoring of the test. He points out that validation should be planned for while still constructing the test. There is no coverage of the procedures, or mechanics, of validation, nor mention of the difficulties and pitfalls involved in attempting to conduct a statistically sound validation study. The author presents a number of specific rules for constructing true-false, multiple choice, completion, and matching test items. He discusses advantages and disadvantages of essay-type tests. He refers to "approaching reasonable objectivity" in scoring essay-type tests without mentioning the extreme difficulty of achieving such objectivity, reasonable or otherwise.

The book contains a short chapter on performance tests. There is no discussion of the cumbersoneness of most performance tests, the time, expense, and difficulty of scoring, when used for most jobs other than those such as office machine operators, typists, and stenographers.

The importance of having new test items reviewed by subject matter experts prior to use is pointed out. A method of item analysis which any test constructor can use is presented.

In conclusion, it should be stated that the book can serve a worthy purpose. It is believed that many potential test constructors who would ordinarily "shy away" from the more technical references on test construction will have a good "starter" in this book and will in turn be introduced to more technical references.

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WILFRID J. DIXON and FRANK J. MASSEY, JR. *An Introduction to Statistical Analysis.* New York: McGraw-Hill, 1951, pp. x + 370. $4.75.

Intended for students in various disciplines who desire a basic course in statistics, the book *Introduction to Statistical Analysis*, represents a unique contribution to the literature of elementary texts in the field. In a lucid style and with a marked degree of simplicity the authors have presented a wealth of material and have logically and coherently developed a vast number of concepts in statistical thinking. Probably the most remarkable feature of the book is that an intelligent student with a background of but three semesters of high school algebra can understand virtually every topic considered. Although authors of elementary texts in mathematical statistics have frequently maintained that little formal mathematical training is required on the part of prospective readers, Dixon and Massey

have probably been the first ones to succeed in their objectives of adapting the content of their subject to the level of mathematical maturity of the majority of college students.

Throughout most of the text the authors have displayed an unusual grasp of the psychology of learning in terms of a common-sense and intuitive approach to the explanation of concepts within the areas of sampling and hypothesis testing—an approach embodying a favorable balance of verbalization, geometric representation, and algebraic expression. A substantial degree of the pedagogical success achieved is probably due to the fact that the authors have followed to a considerable extent the recommendations made by the committee on teaching of statistics of the National Research Council as to the order and emphasis of topics.

Numerous illustrative examples are included that serve to reinforce the expository passages within the text. Particularly appealing is the employment of an outline form with respect to which explanatory problems are worked. In each of the illustrative examples concerning the testing of hypotheses the steps involved are given in logical order and described in a precise manner. Although usually based on ficticious data, the problem exercises at the end of each chapter are thought provoking and essential to mastery of the fundamental concepts developed in the chapters. (Answers are available from the authors upon request.) Several class exercises are included, many of which aim toward the experimental verification of sampling distributions.

For the student of the social sciences the text has much to offer in terms of the attainment of a broader perspective or familiarity with the possible applications of statistics in various fields. Perhaps more important to the social science worker is his exposure to certain intangible features underlying the viewpoint of the mathematical statistician. In reading this book and in teaching a course organized about it, the reviewer has gained the impression of the existence of a certain inherent philosophy in statistical thinking not found in most texts written by statisticians specializing in psychology, education, or sociology. To some extent this feeling may have been due to the emphasis placed by the writers upon such relatively modern developments as α and β errors, power tests, nonparametric statistics, statistical efficiency, and sequential analysis.

One cannot help being impressed with the twenty-six different tables included within the 72 pages of the Appendix. Many of the tables which are not to be found in other texts are based on developments in statistical theory that have appeared in the journals since 1945. At various points in the text examples are included that serve to demonstrate use of the tables.

Only two possible criticisms of the text come to mind. These are based upon reports which the reviewer has received from students. First, students consistently have a great deal of difficulty in understanding how to use the tables of random numbers. The explanation given on page 35 appears to be somewhat confusing, if not incomplete. One or two illustrative examples at this point would probably be helpful. Second, several students—especially those with limited mathematical maturity—state that they find it hard to grasp the central implications of α and β errors developed in Chapter 7 and expanded upon in Chapter 14. In particular, the explanation regarding the power of a test and the power of a critical region against certain alternatives does not satisfy many students. Although the authors' exposition appears to be quite explicit to the reviewer, it may be that the incorporation of additional diagrams involving overlapping geometric curves with appropriate designations of α and β in terms of shaded areas or the insertion of two or three more illustrative examples would be helpful. Several students have mentioned that it was not until they studied the first two or three pages of the Chapter 18 "Sequential Analysis" that they began to grasp the import of α and β errors and to comprehend the rôle of power tests associated with these errors.

In addition to the topics cited that are based upon relatively recent developments

such conventional subjects as analysis of variance, regression and correlation, analysis of covariance, and enumeration statistics are considered. For the psychology student the content of the chapter concerning regression and correlation is probably inadequate, since discussion is limited to a consideration of two-variable correlation. However, it is not reasonable to expect that an introductory text can cover adequately specific points of emphasis in the various fields to which statistical methods are applied.

In any event, the student in the social sciences who is seriously interested in understanding and applying the fundamentals of statistics will do well to read this book. In fact, the social scientist who wishes a reference text on a variety of topics within the area of statistics should by all means purchase this book for his library, as he will find it to be a convenient working aid and an excellent source for periodic review.

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